CSE 107 1-11-24

Supplemental Lecture

Observe

- $P(A|B) \cdot P(B) = \underbrace{P(A \cap B) = P(A) P(B|A)}_{\text{generalize this}}$

- $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$ ✓

Prove this:

$$RHS = \cancel{P(A)} \cdot \frac{\cancel{P(A \cap B)}}{\cancel{P(A)}} \cdot \frac{P(A \cap B \cap C)}{\cancel{P(A \cap B)}}$$

$$= P(A \cap B \cap C) = LHS$$

- $P(A \cap B \cap C \cap D)$

$$= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \cdot P(D|A \cap B \cap C)$$

Exercise: Prove this

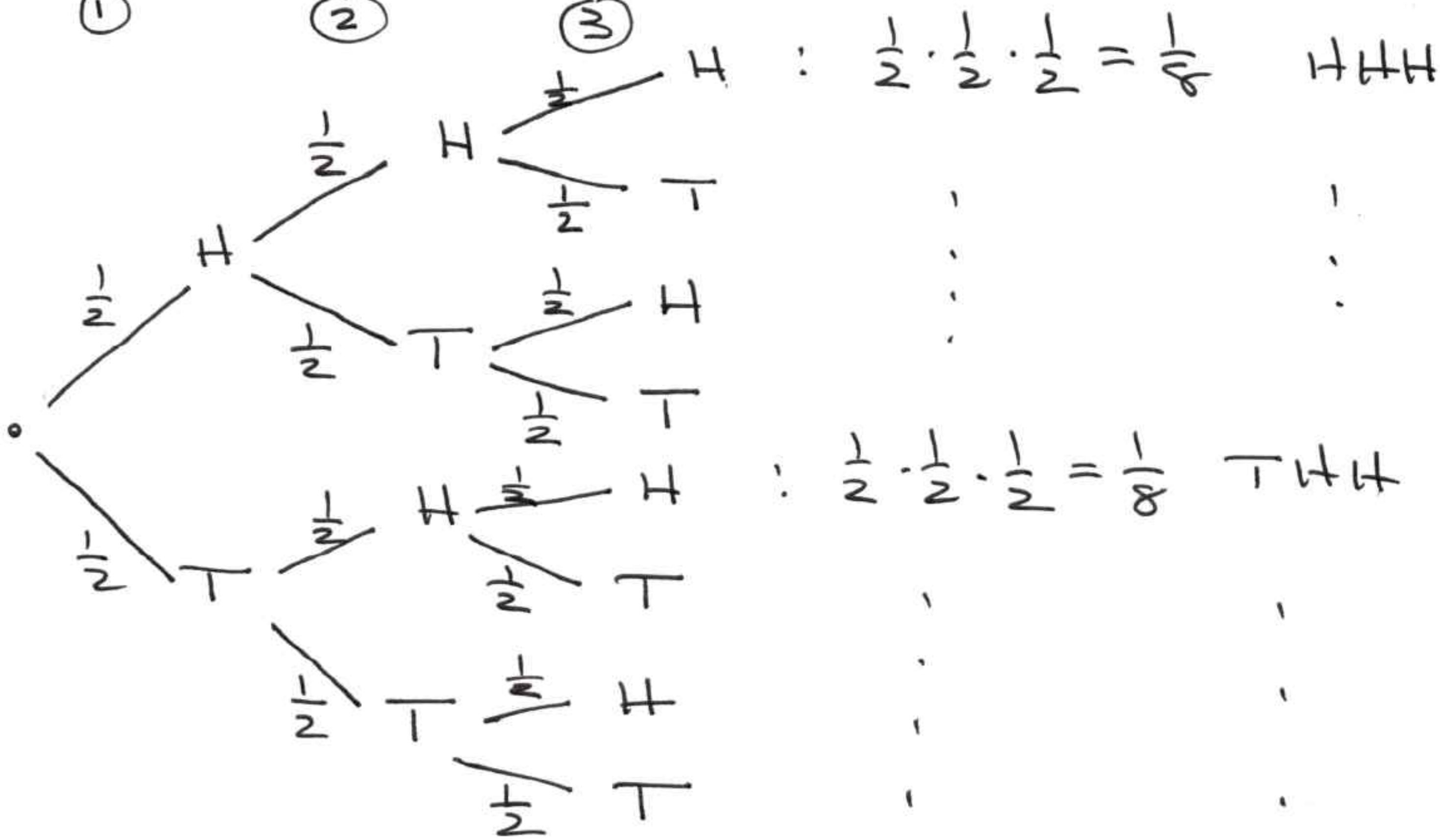
Multiplication Rule

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots$$

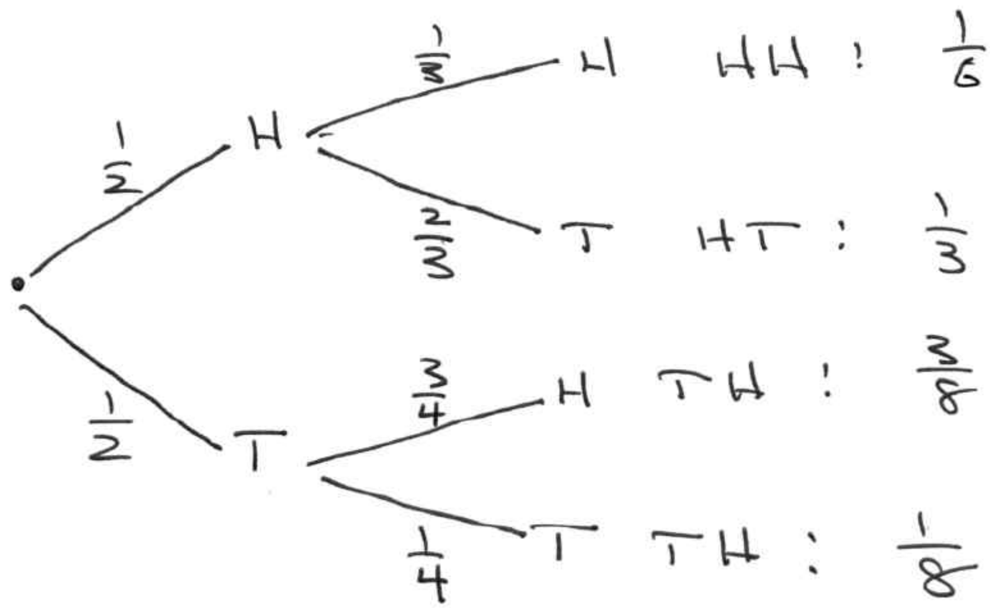
$$\cdots P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right)$$

Common Practice: multiply Prob. down a tree diagram.

EX Toss 3 fair coins

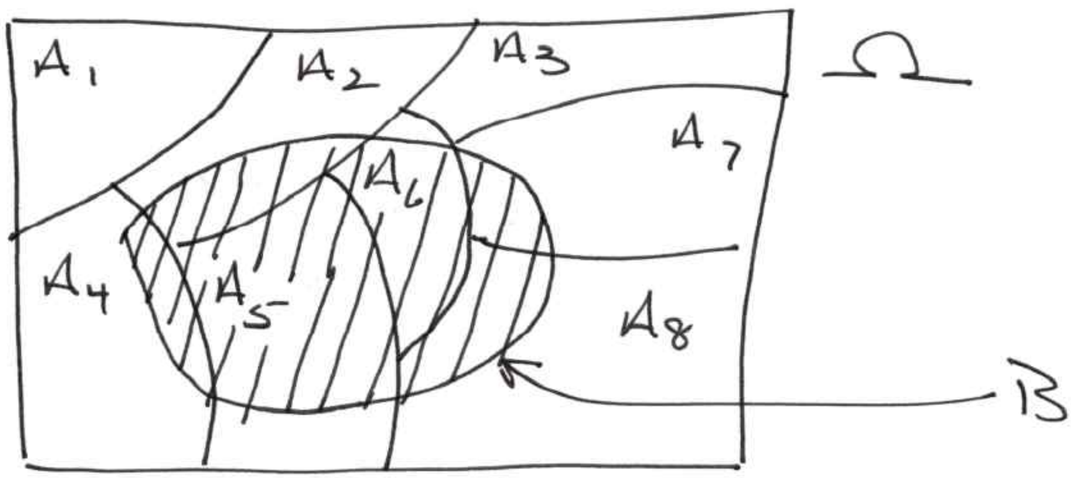


EX. 3 coins {1, 2, 3}: flip 1, if head, flip 2, if tails, flip 3



1.4 Total Probability, Bayes rule

Let $A_1, A_2, \ldots, A_n \subseteq \Omega$ be pairwise disjoint (i.e. $A_i \cap A_j = \phi$ if $i \neq j$). Assume $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$. We call this a partition of Ω.



Total Probability Theorem

Assume $P(A_i) > 0$ for each $1 \le i \le n$. Then for any $B \subseteq \Omega$ we have

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(B \cap A_n)$$

$$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \cdots + P(A_n) \cdot P(B|A_n)$$

Ex. Roll a fair die. If result is in $\{1, 2, 3\}$, roll again, otherwise stop. What is the prob. of

$$B = \{\text{sum of all rolls} \geq 5\}$$

Define

$$A_i = \{1^{st} \text{ roll is } i\} \quad (1 \leq i \leq 6).$$

$\longrightarrow P(A_i) = \frac{1}{6}$

Note

if A_1 occurs, then B occurs iff $2^{nd} \in \{4, 5, 6\}$
" A_2 " " " " " " $\in \{3, 4, 5, 6\}$
" A_3 " " " " " " $\in \{2, 3, 4, 5, 6\}$

Thus

$$P(B|A_1) = \frac{3}{6} = \frac{1}{2}$$
$$P(B|A_2) = 4/6 = 2/3$$
$$P(B|A_3) = 5/6$$
$$P(B|A_4) = 0$$
$$P(B|A_5) = 1$$
$$P(B|A_6) = 1$$

$$P(B) = P(B|A_1)\cdot P(A_1) + \cdots + P(B|A_6)\cdot P(A_6)$$

$$= \frac{1}{2}\cdot\frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 1\cdot\frac{1}{6} + 1\cdot\frac{1}{6}$$

$$= \frac{1}{6}\left(\frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 0 + 1 + 1\right) = \boxed{\frac{2}{3}}$$

Another Way

$$\Omega = \begin{array}{cccccc} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 4 & & & & & \\ 5 & & & & & \\ 6 & & & & & \end{array}$$

$B = \{sum \geq 5\}$

$$P(B) = \frac{12}{36} + \frac{2}{6} = \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

Baye's Rule

Let $A_1, \ldots, A_n$ be pairwise disjoint events forming a partition of Ω. Assume $P(A_i) > 0$ for $i = 1, \ldots, n$. Also assume $B \subseteq \Omega$ with $P(B) > 0$. Then for each $i = 1, 2, \ldots n$

$$P(A_i | B) = \frac{P(A_i) \cdot P(B | A_i)}{P(B)}$$

$$= \frac{P(A_i) \cdot P(B | A_i)}{P(A_1) \cdot P(B | A_1) + \cdots\cdots + P(A_n) \cdot P(B | A_n)}$$

<u>Proof</u>: follows from defn of cond. Prob. and total Prob. thm. ∎

Bayes' rule is used for 'inference' Problems.

$$\begin{array}{cccl} A_1 & \searrow & & \\ A_2 & \longrightarrow & B & : \underbrace{P(B|A_i)}_{\substack{\text{measure of}\\ \text{strength of}\\ \text{causation.}}} \\ \vdots & \nearrow & & \\ A_n & & & \\ \text{causes} & & \text{effect} & \end{array}$$

we go from effect to cause

$$A_i \xleftarrow[\text{inference}]{\qquad\qquad} B \; : \; \underbrace{P(A_i \mid B)}_{\substack{\text{measure of}\\ \text{srength of}\\ \text{inference}}}$$

$$\underbrace{P(A_i \mid B)}_{\text{Posterior}} = \frac{\overbrace{P(A_i)}^{\text{Prior}}\,\overbrace{P(B \mid A_i)}^{\text{likelihood}}}{\underbrace{P(B)}_{\text{marginal}}}$$

often we use Partition: $A \cup A^c = \Omega$

Then Baye's rule Becomes:

$$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{P(B)}$$

$$= \frac{P(A) \cdot P(B \mid A)}{P(A) \cdot P(B \mid A) + P(A^c) \cdot P(B \mid A^c)}$$

Ex. disease example again

$T = \{\text{test positive}\}$

$D = \{\text{has disease}\}$

We were given

$P(T|D) = .99$

$P(T|D^c) = .10$

$P(D) = .05 \quad \therefore P(D^c) = .95$

To find

$$P(D|T) = \frac{P(D)\cdot P(T|D)}{P(T)}$$

$$= \frac{P(D)\cdot P(T|D)}{P(D)\cdot P(T|D) + P(D^c)\cdot P(T|D^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99) + (.95)(.10)} = \boxed{.3426}$$

round to 4