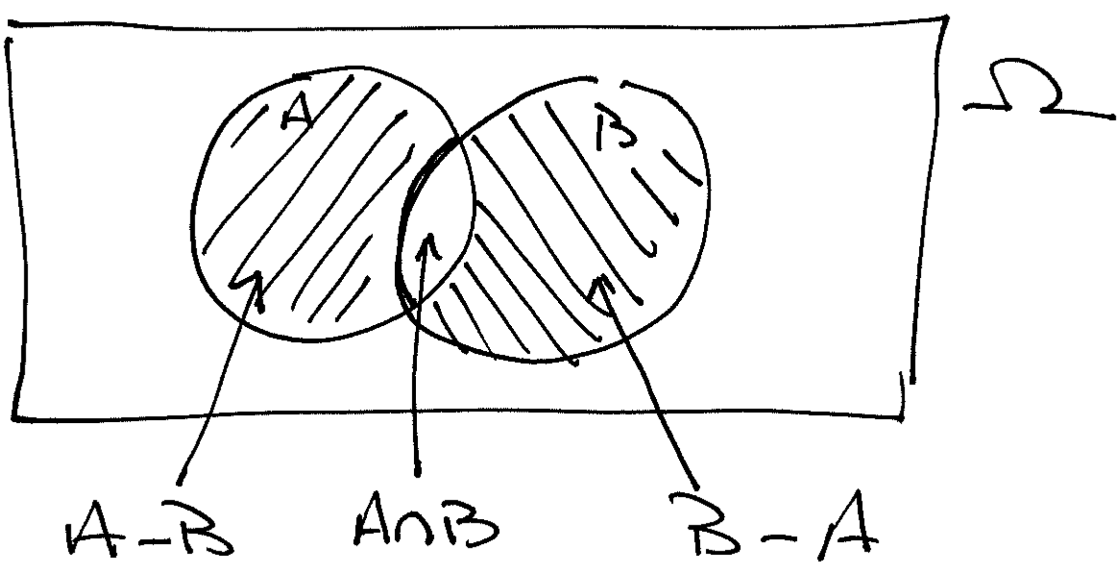
(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof



$$P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$$

Also

$$P(B) = P(B-A) + P(A\cap B)$$

$$\therefore \quad P(B-A) = P(B) - P(A\cap B)$$

similarly

$$P(A-B) = P(A) - P(A\cap B)$$

Thus

$$P(A\cup B) = \left(P(B) - \cancel{P(A\cap B)}\right) + \cancel{P(A\cap B)} + \left(P(A) - P(A\cap B)\right)$$

$$= P(A) + P(B) - P(A\cap B).$$

$\blacksquare$

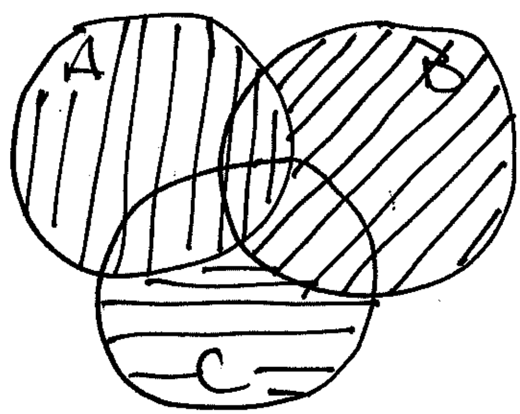
(c) $P(A \cup B) \le P(A) + P(B)$

follows from (b)

(d) $P(A \cup B \cup C)$

$= P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

Proof



note:

(1) $A,\ A^c \cap B,\ A^c \cap B^c \cap C$

are Pairwise disjoint.

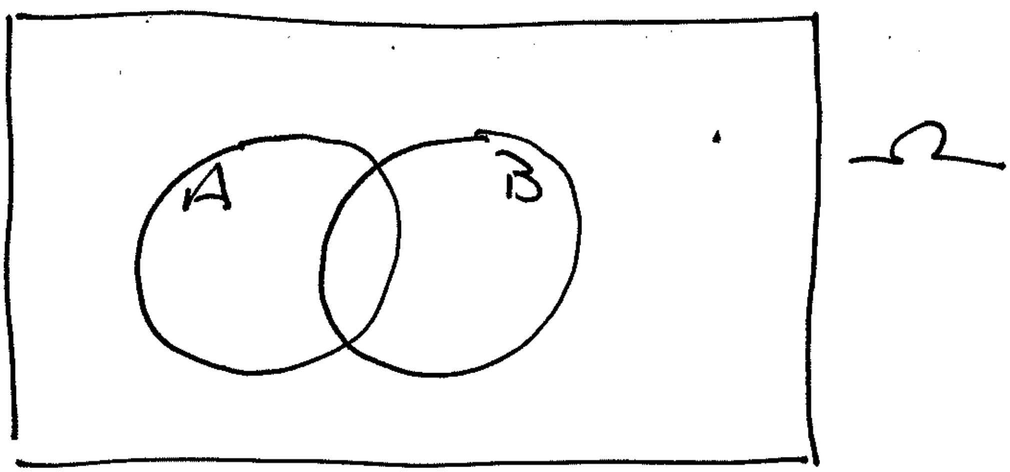
and

(2) $A \cup B \cup C$ is their union

Result follows from axiom iii. ∎

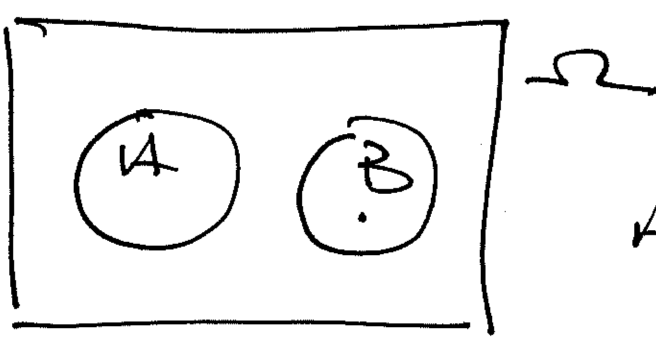
1.3 Conditional Probability

let $A, B \subseteq \Omega$



Two extreme cases

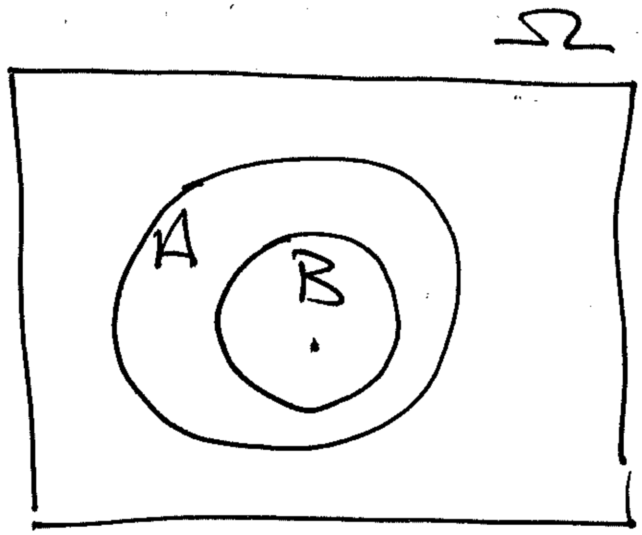
(1)



$A \cap B = \phi$

B occurs $\Longrightarrow$ A <u>does not</u> occur

(2)



$B \subseteq A$

B occurs $\Rightarrow$ A does occur

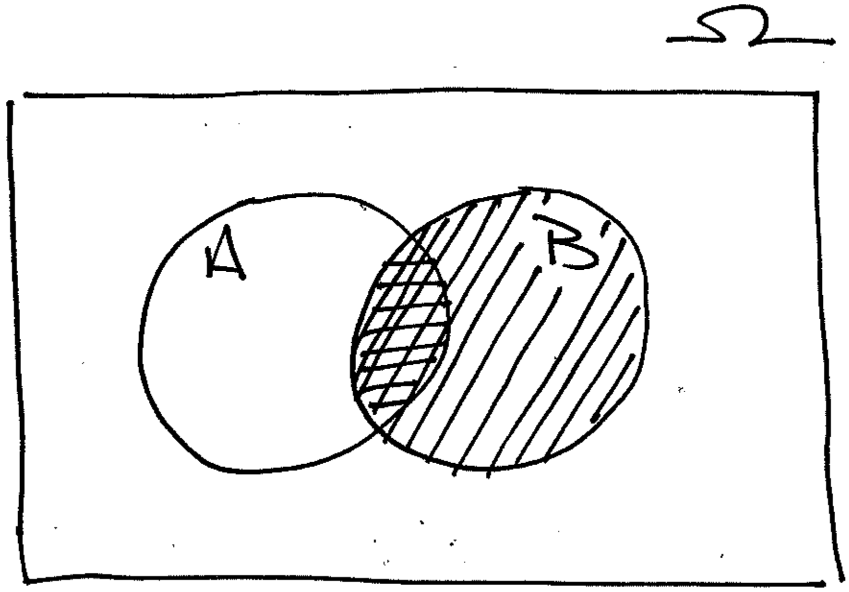
Defn

The conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

(we assume $P(B) > 0$).

Picture



observe that $P(\cdot \mid B)$ is a valid probability law, in that it satisfies the axioms:

(i) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \geq 0$

(ii) $P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(iii) if $A_1 \cap A_2 = \phi$, then

$$P(A_1 \cup A_2 | B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 | B) + P(A_2 | B).$$ $\blacksquare$

It follows that <u>all</u> theorems and identities that we've proved from axioms, also hold for $P(\cdot \mid B)$

For example: for <u>any</u> $A_1, A_2 \subseteq \Omega$

- $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

<u>Exercise</u>: Prove this!!

Also, if Ω is finite with all elements equally likely, then

- $$P(A|B) = \frac{|A \cap B|}{|B|}$$

<u>Exercise</u>: Prove this also

<u>Note</u>:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Given: $P(A \cap B) = P(A|B) \cdot P(B)$

and: $P(A \cap B) = P(B|A) \cdot P(A)$

Ex.

A test for a disease is Positive with Prob. .99 if the subject has the disease, and with Prob .10 if subject doesnt have it. The Prob. that a random Person has disease is .05

- What is Prob. of false Positive?
- What " " " " Negative?

Let

$T = \{\text{test is Positive}\}$

$D = \{\text{subject has disease}\}$

we are given

$P(T|D) = .99 \quad \therefore \quad P(T^c|D) = .01$

$P(T|D^c) = .10 \quad \therefore \quad P(T^c|D^c) = .90$

also $P(D) = .05$, $\therefore \quad P(D^c) = .95$

we want

$P(T \cap D^c) = P(T|D^c) \cdot P(D^c)$

$= (.10) \cdot (.95) = \boxed{.095}$

and

$$P(T^c \cap D) = P(T^c | D) \cdot P(D)$$

$$= (.01)(.05) = \boxed{.0005}$$