CSE 107
Homework Assignment 7

1. Let X be exponentially distributed with parameter $\lambda = 1$, and let Y be uniformly distributed over the interval $[0, 1]$. Use convolution to find the PDF of $Z = X - Y$.

2. Let X be a discrete random variable with PMF $p_X(x)$, and let Y be a continuous random variable, independent from X, with PDF $f_Y(y)$. Derive a formula for the PDF of the continuous random variable $Z = X + Y$.

Section 4.3: Conditional Expectation and Variance

3. Let R be a continuous uniform random variable on the interval $[1/n,\ 1]$, and let X be a geometric random variable whose parameter is itself the random variable R. Find $E[X]$ and $\lim_{n\to\infty} E[X]$. (Hint: use the law of iterated expectations: $E[X] = E\big[E[X|R]\big]$.)

Section 4.5: The Sum of a Random Number of Random Variables

4. Given two fair 6-sided dice and a fair coin:

    a. Roll one die, then flip the coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained.

    b. Roll both dice, observe the sum of the faces showing, then flip the coin the number of times given by the sum. Find the expected value and variance of the number of heads obtained.

Section 6.1: The Bernoulli Process

5. Consider a Bernoulli process with parameter p.

    a. Let T be the (discrete) time until the first arrival. Write down the PMF of T.

    b. Let N_t be the number of arrivals in the time interval $[1, t]$ (where t is a positive integer). Write down the PMF of N_t.

    c. Let Y_k be the time until the k^{th} arrival. Write down the PMF of Y_k.

    d. Let H be the number of arrivals that occur in the time interval $[1, 20]$, and let C be the event that exactly 10 arrivals occur in the time interval $[1, 18]$. Determine the conditional expectation $E[H|C]$ and the conditional variance $\text{Var}(H|C)$. (Hint: each number depends on p.)

    e. Now suppose there is a second, independent Bernoulli process, also with parameter p, running parallel to the first (i.e. running on the same discrete time clock.) Consider the merged process in which an arrival occurs whenever there is an arrival in the first process, or in the second, or in both. Determine the probability that the 5$^{\text{th}}$ arrival occurs at time 10.

6. Let Y_{17} be the Pascal random variable of order 17 and parameter p. Determine positive integers a and b such that

$$\sum_{j=42}^{\infty} p_{Y_{17}}(j) = \sum_{k=0}^{a} \binom{b}{k} p^k (1-p)^{b-k}$$

(Hint: consider a Bernoulli process of parameter p, and identify both sides of the above equation as the probability of one and the same event.)