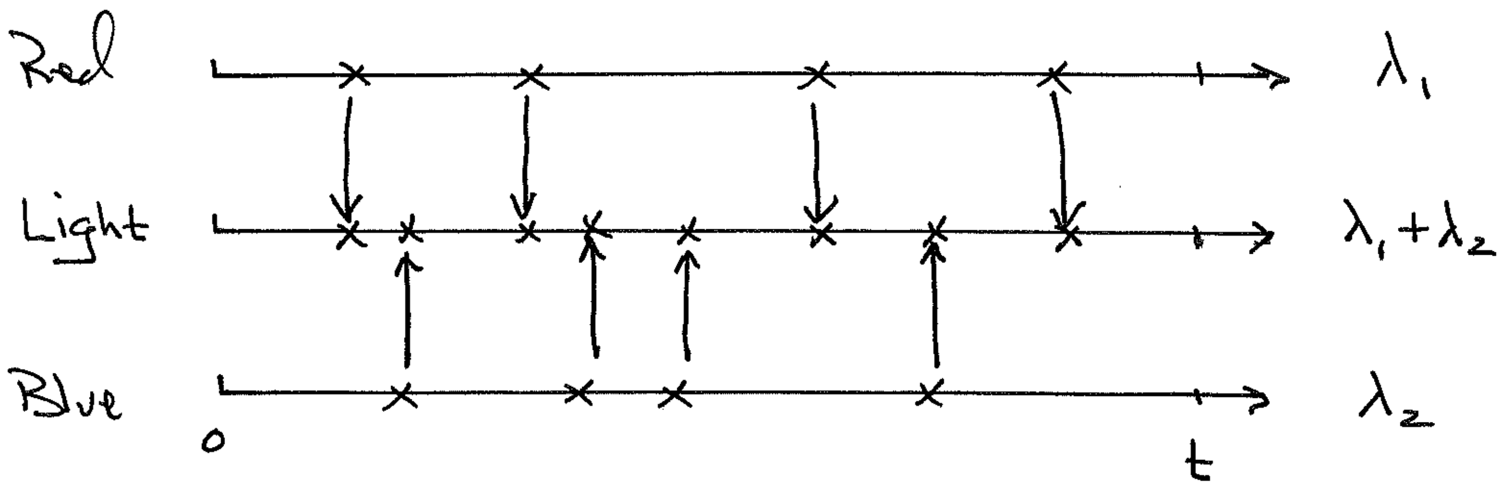
CSE 107 3-8-24

Supplemental Lecture

Merging Poisson Processes

Ex.
An observer sees 2 lights that flash at random times, one Red one Blue. The Red is a Poisson Process with rate λ_1, the Blue is Poisson with rate λ_2. The observer is colorblind and sees just flashing lights.

What kind of Process is the observer seeing?



Let

X = # Red lights in $[0,t]$

Y = # Blue " " "

M = # lights " "

we know X is a Poisson r.v. with parameter $\lambda_1 t$, and same for Y but with parameter $\lambda_2 t$

we have $M = X+Y$, hence M is a Poisson r.v. with

Parameter

$$\lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2) t$$

$\therefore$ the Process seen by the observer is Poisson with rate $\lambda_1 + \lambda_2$.

Ex.

suppose observer sees a single flash. what is the Probability that it was Red ?

Solution !

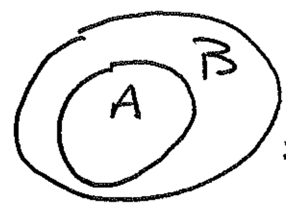
use small interval Probabilities.

Let δ be the width of a small time interval containing the flash.

We seek

$$P(1 \text{ Red} \mid 1 \text{ light})$$

$$= \frac{P(1 \text{ Red}, 1 \text{ light})}{P(1 \text{ light})}$$



$A \subseteq B$

$\Rightarrow A \cap B = A$

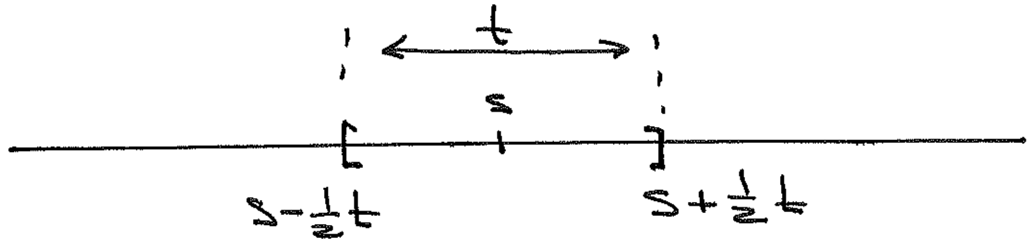
$$= \frac{P(1 \text{ Red})}{P(1 \text{ Light})}$$

$$= \frac{\lambda_1 \delta}{(\lambda_1 + \lambda_2)\delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

<u>solution 2</u>

without small interval probabilities.

suppose light flashes at time s



consider an interval of width t centered at s. $[s-\frac{1}{2}t, s+\frac{1}{2}t] = I_t$

We seek

$$P(1 \text{ Red in } I_t \mid 1 \text{ light in } I_t)$$

$$= \frac{P(1 \text{ Red in } I_t ; 1 \text{ light in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \frac{(\lambda_1+\lambda_2)t}{1!}}$$

$$= e^{\lambda_2 t} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} \xrightarrow[t \to 0]{} \frac{\lambda_1}{\lambda_1+\lambda_2}$$

as $t \to 0$ ∎

7.1 Discrete Markov Chains

We consider now processes in which future does depend on the past. That dependence is embodied by a <u>state</u> which changes over time. We always take set of states S to be

$$S = \{1, 2, 3, \ldots, m\} .$$

Let $X_0, X_1, X_2, \ldots, X_n, \ldots$ be a sequence of r.v.s taking values in S. We consider

$$X_n = \text{the state at time } n .$$

we define <u>transition probabilities</u> p_{ij} to be

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

for any $i, j \in S$ and $n \geq 0$. This process satisfies the <u>Markov Property</u> iff

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= P(X_{n+1} = j \mid X_n = i) = p_{ij}$$

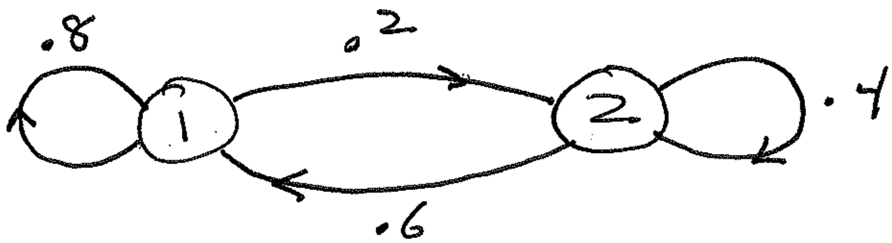
i.e. the next state depends on the past only through the immediately preceding state, not the entire history.

Defn

A <u>Markov chain model</u> is a random process on state space S and satisfying Markov Property.

Ex. $m=2$, $S=\{1,2\}$ and

$P_{11}=.8$, $P_{12}=.2$, $P_{21}=.6$, $P_{22}=.4$



State Transition Diagram

Suppose $\boxed{X_0=1}$. Find $P(X_2=2)$

$$P(X_2=2)=P(X_2=2\mid X_1=1)\cdot P(X_1=1)+P(X_2=2\mid X_1=2)\cdot P(X_1=2)$$

By total Prob. thm.

$$= P_{12} \cdot \underbrace{P(X_1 = 1)}_{P_{11}} + P_{22} \underbrace{P(X_1 = 2)}_{P_{12}}$$

note:

$$P(X_1=1) = P(X_1=1 \mid X_0=1) \cdot \overbrace{P(X_0=1)}^{1}$$

$$+ P(X_1=1 \mid X_0=2) \cdot \overbrace{P(X_0=2)}^{0}$$

$$= P_{11}$$

Also $P(X_1=2) = P(X_1=2 \mid X_0=1) = P_{12}$

Thus

$$P(X_2=2) = P_{12} \cdot P_{11} + P_{22} \cdot P_{12} = P_{11} P_{12} + P_{12} P_{22}$$

$$= (.8)(.2) + (.2)(.4)$$

$$= .16 + .08 = \boxed{.24}$$

Similarly

$$P(X_2=1) = P_{11} \cdot P_{11} + P_{12} P_{21} = (.8)^2 + (.2)(.6) = \boxed{.76}$$

In the same manner, if $\boxed{X_0 = 2}$,

then

$$P(X_2=1) = P_{21}P_{11} + P_{22}P_{21} = \boxed{.72}$$

and

$$P(X_2=2) = P_{21}P_{12} + P_{22}P_{22} = \boxed{.28}$$

we can summarize all these calculations using <u>transition probability matrix</u>.

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

called a probability matrix.

observe

$$R^2 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$= \begin{pmatrix} P_{11}^2 + P_{12}P_{21} & P_{11}P_{12} + P_{12}P_{22} \\ P_{21}P_{11} + P_{22}P_{21} & P_{21}P_{12} + P_{22}^2 \end{pmatrix}$$

$$= \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}$$

Thus the ij^{th} element of R^2 is the Probability of going from state i to state j in exactly 2 steps.

Defn

The <u>n-step transition probabilities</u> are

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

for any $i, j \in S$ and $n \geq 0$. note $p_{ij} = r_{ij}(1)$. Thus $r_{ij}(n)$ is Prob. of going from state i to state j in n steps.

<u>Theorem</u> Chapman-Kolmogorov equations.

$$\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj} \qquad (i, j \in S,\ n \geq 2)}$$