CSE 107 3-7-24

Recall:

$P(k \text{ arrivals in } [0,t])$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

Let N_t denote the # of arrivals of the Poisson Process in $[0,t]$ (rate λ). Then

$$P_{N_t}(k) = P(N_t = k) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

$(k = 0, 1, 2, \ldots)$

This is the PMF of a Poisson r.v. with parameter λt.

note also

$$E[N_t] = \lambda t, \quad Var(N_t) = \lambda t.$$

Let T be time to 1st arrival.

observe

$$\{T > t\} = \{N_t = 0\}$$

Thus

$$F_T(t) = P(T \leq t)$$

$$= 1 - P(T > t)$$

$$= 1 - P(N_t = 0)$$

$$= 1 - p_{N_t}(0)$$

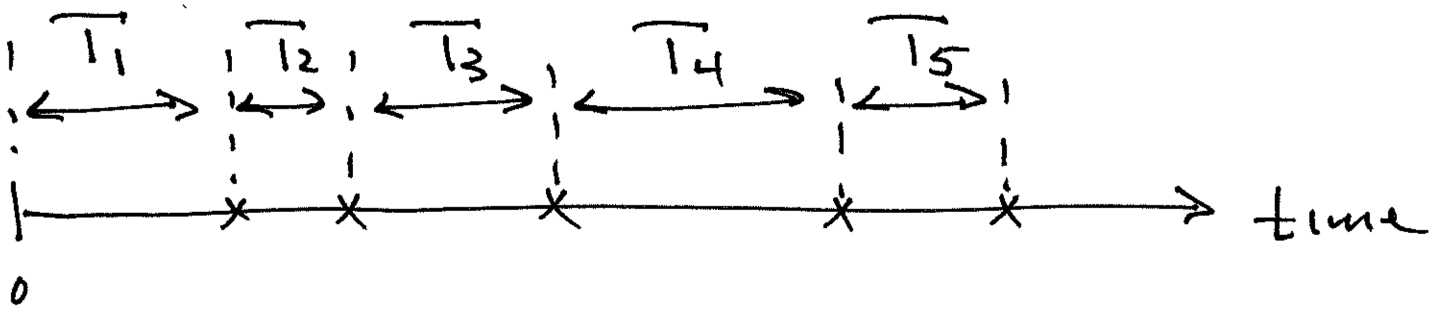
$$= 1 - e^{-\lambda t}$$

Differentiate w.r.t t to get

$$f_T(t) = \lambda e^{-\lambda t} \qquad (t \geq 0)$$

which is exponential with parameter λ.

Let T_K be the K^{th} inter-arrival time $(K = 1, 2, 3 \ldots)$



By the 'fresh start' or 'memoryless' Property, each T_K is also exponential (λ)

$$f_{T_K}(t) = \lambda e^{-\lambda t} \qquad (t \geq 0).$$

Now define

$$Y_K = T_1 + T_2 + \cdots + T_K$$

$$(K = 1, 2, \ldots)$$

so Y_k is time to k^{th} arrival.

The dist. of Y_k is called Erlang of order k. (derived in book p. 316-317.)

Exercise

Prove that sum of k independent exponential (λ) r.v. is Erlang of order k.

Hint: use induction on k. When $k=1$ Erlang is exponential, use convolution on induction step to show

$$Y_K = Y_{K-1} + T_K$$

is Erlang of order K.

Summarize: Bernoulli vs. Poisson

	Bernoulli	Poisson
time:	$t \in \{1, 2, 3, \dots\}$	$t \in [0, \infty)$
rate:	$p \ \frac{\text{arrivals}}{\text{trial}}$	$\lambda \ \frac{\text{arrivals}}{\text{unit time}}$
N_t:	Binomial(t, p)	Poisson(λt)
T_K:	Geometric(p)	Exponential(λ)
Y_K:	Pascal(p, K)	Erlang(λ, K)

Ex.

You receive email by a Poisson Process at rate $\lambda = \dfrac{2 \text{ messages}}{\text{hour}}$

You currently have no new messages.

a.) What is the Probability of finding 3 new messages when you check in 30 min.?

$$P\left(N_{\frac{1}{2}} = 3\right) = P_{N_{\frac{1}{2}}}(3)$$

$$= e^{-(2\cdot\frac{1}{2})} \cdot \frac{(2\cdot\frac{1}{2})^3}{3!}$$

$$= e^{-1} \cdot \frac{1}{6} = \boxed{.0613}$$

b.) what is prob. of no new messages in 30 min.?

$$P\left(N_{\frac{1}{2}}=0\right)=P_{N_{\frac{1}{2}}}(0)$$

$$=e^{-(2\cdot\frac{1}{2})}\cdot\frac{(2\cdot\frac{1}{2})^0}{0!}$$

$$=e^{-1}=\boxed{.3679}$$

c.) what is expected time to 5^{th} new message.

$$E[Y_5]=E[T_1+\cdots+T_5]$$

$$=E[T_1]+\cdots+E[T_5]$$

$$=5\cdot\frac{1}{2}=\frac{5}{2}=\boxed{2.5 \text{ hours}}$$

The Sum of Poisson r.v.s

Let X, Y be Poisson r.v.s with parameters λ_1, λ_2, resp.

Then

$$Z = X + Y$$

is Poisson with parameter $\lambda_1 + \lambda_2$.

<u>Proof</u> we have

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \quad (k = 0, 1, 2, \ldots)$$

and

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \quad (k = 0, 1, \ldots)$$

Thus

$$P_Z(n) = \sum_{k=0}^{\infty} P_X(k) \cdot P_Y(n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{(n-k)}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \lambda_1^k \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot (\lambda_1+\lambda_2)^n$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!}$$

$\therefore$ Z is Poisson with Parameter $\lambda_1 + \lambda_2$.

<u>EX</u>

During rush hour (8-9 am), accidents occur as a Poisson Process with rate 5/hour.

During 9-11 am, accidents occur as a Poisson Process with rate 3/hour.

What is the PMF of the total # of accidents in 8-11 am?

solution

let $N_1 = \#$ accidents in 8-9 am

$N_2 = \#$ accidents in 9-11 am

and so

$N = N_1 + N_2 = \#$ accidents in 8-11.

we know N_1 is Poisson $(\lambda_1 = 1 \cdot 5 = 5)$

N_2 " " $(\lambda_2 = 2 \cdot 3 = 6)$

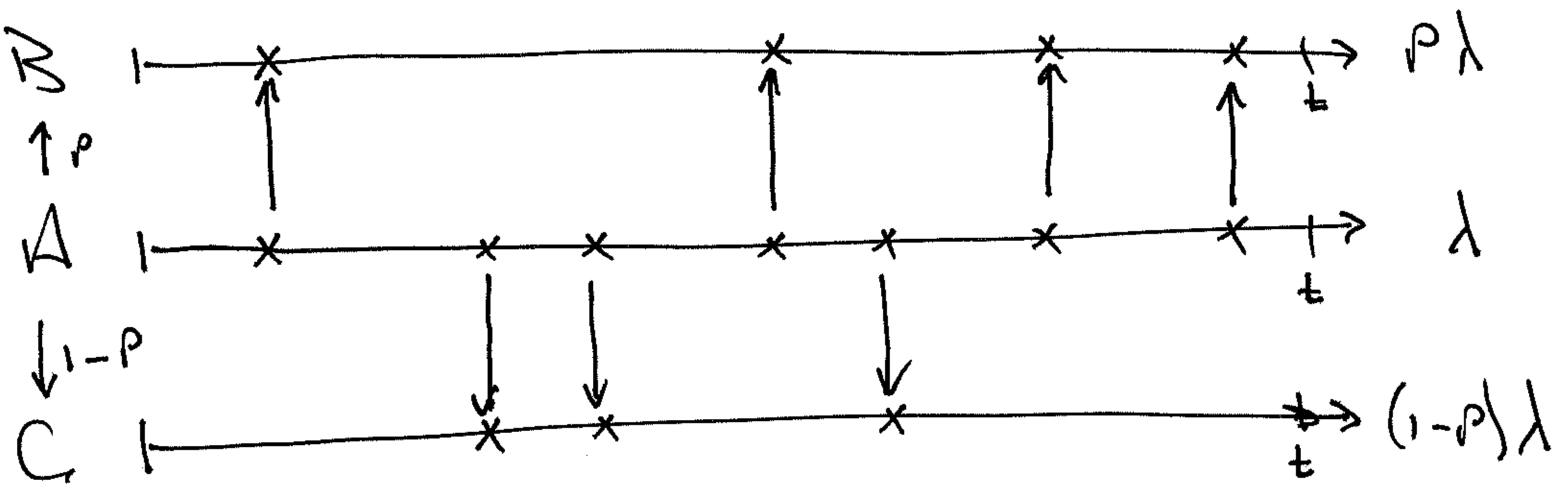
$\therefore$ N is Poisson $(\lambda = \lambda_1 + \lambda_2 = 11)$

Thus

$$P_N(n) = e^{-11} \cdot \frac{11^n}{n!}$$

Splitting a Poisson Process

Let A be a Poisson Process of rate λ. We split A into two arrival Processes B and C, with Probabilities p & $(1-p)$.



What kinds of Processes are B and C?

Let N = # arrivals in A in $[0, t]$

X = " " " B " "

Y = " " " C " "

We know N is Poisson(λt), so

$$P_N(n) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \quad (n = 0, 1, 2, \ldots)$$

Then

$$P_X(k) = P(X = k)$$

$$= \sum_{n=k}^{\infty} P(X = k \mid N = n) \cdot P(N = n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=K}^{\infty} \frac{\cancel{n!}}{K!\,(n-K)!} \cdot p^K (1-p)^{n-K} e^{-\lambda t} \cdot \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^K}{K!} \cdot e^{-\lambda t} \cdot \sum_{n=K}^{\infty} \frac{(1-p)^{n-K} \cdot (\lambda t)^n}{(n-K)!}$$

$$\{\text{do sub: } n \leftarrow n+K\}$$

$$= \frac{p^K}{K!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n (\lambda t)^{n+K}}{n!}$$

$$= \frac{(p\lambda t)^K}{K!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p)\lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^K}{K!} e^{-\lambda t} \cdot e^{(1-p)\lambda t}$$

$$= \frac{(p\lambda t)^K}{K!} e^{-\cancel{\lambda t} + \cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!}$$

$\therefore$ X is Poisson$(p\lambda t)$

$\therefore$ B is a Poisson Process with rate $p\lambda$.