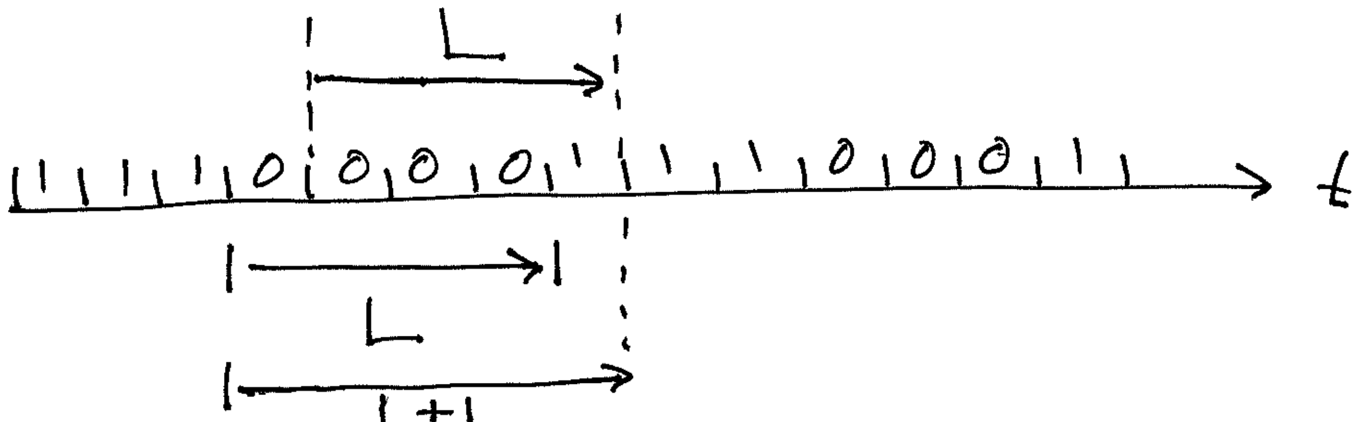
CSE 107 3-5-24

Ex.

$P(\text{win}) = p$

You buy a lottery ticket every day.
Let L be the length of your 1^{st} losing streak.



What is the distribution of L?

note: L cannot be 0, so $L+1$ cannot be 1, so $L+1$ is not geometric.

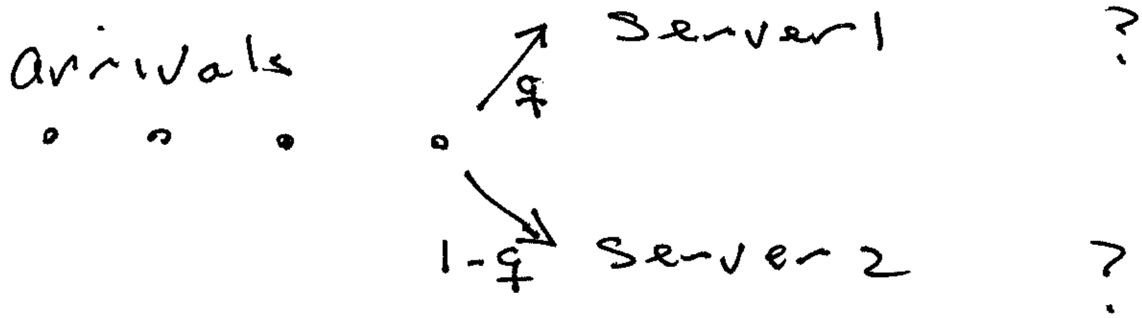
L is also the time from 2nd loss to next win, so L is geometric with param p

splitting a Bernoulli Process

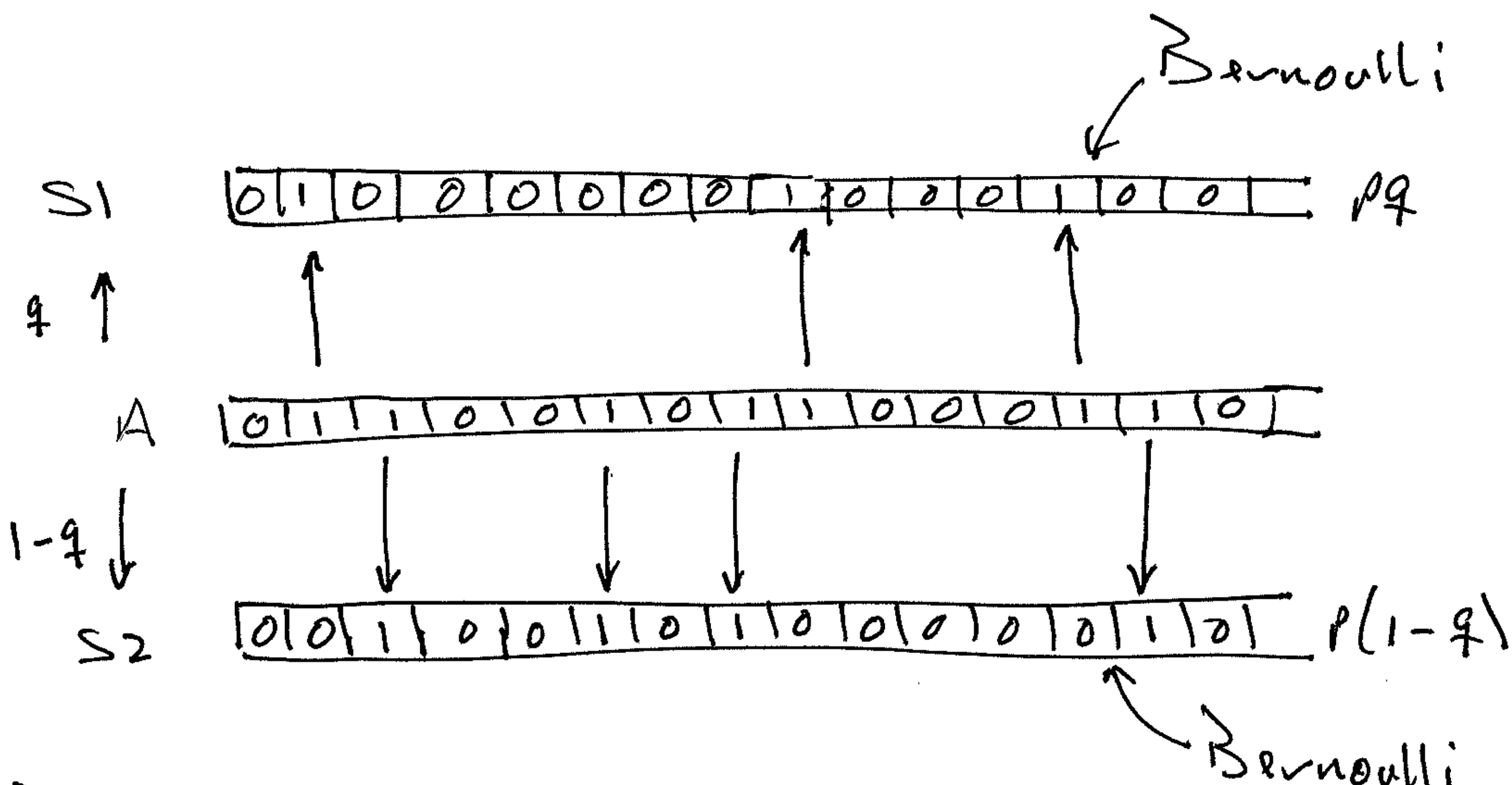
say we have a Bernoulli process of param p, producing a stream of jobs into a queue. each job goes to one of 2 servers

$P(\text{server } 1) = q$

$P(\text{server } 2) = 1 - q$



what kind of processes are seen by S1 and S2 ?



$P(\text{arrival in } S1) = p \cdot q$

$P(\text{arrival in } S2) = p \cdot (1-q)$

Merging Bernoulli Processes

We have independent, simultaneous Bernoulli Processes.

		Params
stream 1 :	$X_1, X_2 \dots$	p
stream 2 :	$Y_1, Y_2 \dots$	q

merge S1 and S2 into a new Processes M

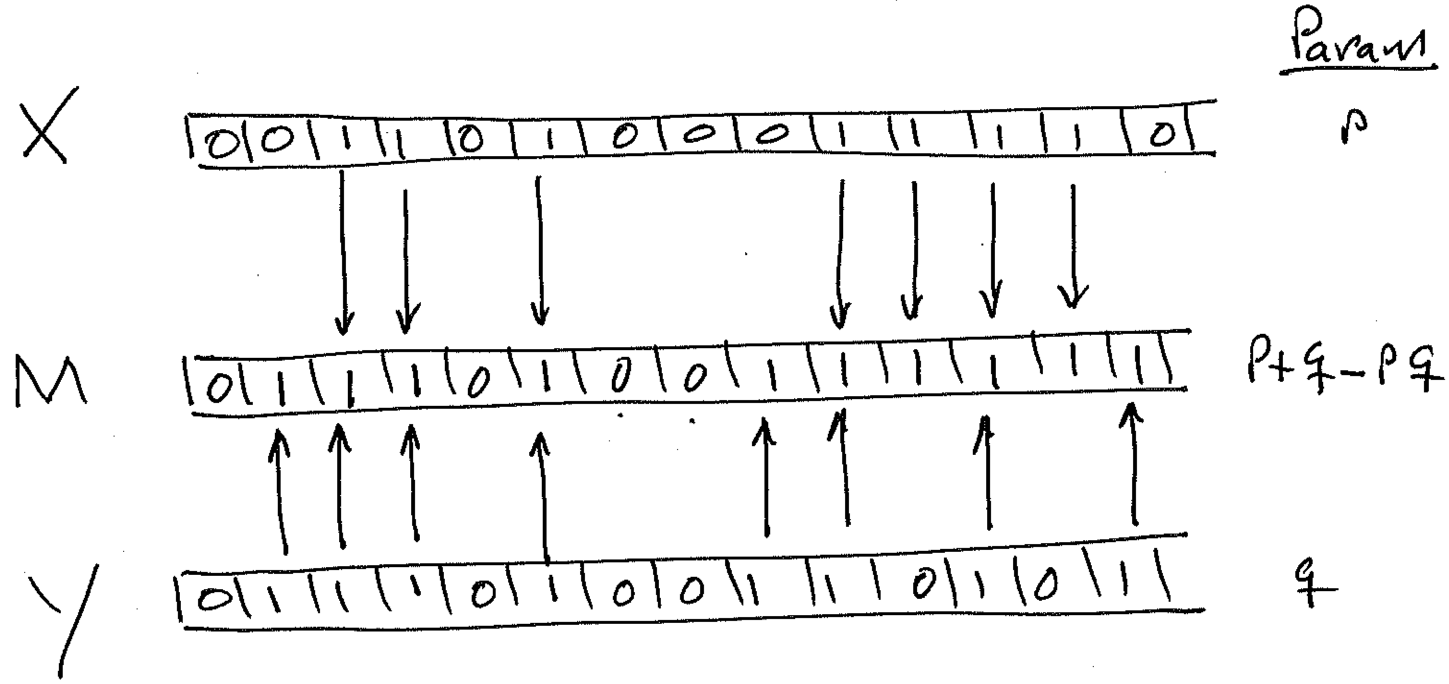
merge : $M_1, M_2, \dots$?

What kind of arrival process is M ?

merge according to the rule

$$M_t = \begin{cases} 1 & \text{if either } X_t = 1 \text{ or } Y_t = 1 \\ 0 & \text{otherwise} \end{cases}$$

inclusive or



$$P(M_t = 1) = 1 - P(M_t = 0)$$

$$= 1 - P(X_t = 0, Y_t = 0)$$

$$= 1 - P(X_t = 0) \cdot P(Y_t = 0)$$

$$= 1 - (1-p)(1-q)$$

$$= 1 - (1 - p - q + pq)$$

$$= p + q - pq$$

note: M is Bernoulli.

Exercises

- 1st split, then merge. find the parameter of resulting B.P.

- first merge, then split
  find parameters of B.P.s

- change bit operation in
  merge from (inclusive) or
  to: <u>and</u>, <u>exor</u>, <u>nand</u>, <u>nor</u>.

6.2 The Poisson Process

Recall: a Bernoulli Process is an infinite seq.

$$X_1, X_2, X_3, \ldots$$

where

(1) each X_t is Bernoulli(p).

(2) the set $\{X_t \mid t = 1, 2, \ldots\}$ is independent, i.e. any finite set is independent!

$$P(X_5 = 0, X_{11} = 1, X_{44} = 1)$$
$$= P(X_5 = 0) \cdot P(X_{11} = 1) \cdot P(X_{44} = 1).$$

we defined

$$Y_k = \text{time to } k^{th} \text{ arrival.}$$

found Y_k has Pascal distribution of order k. also defined

$$T_1 = Y_1$$

$$T_2 = Y_2 - Y_1$$

$$\vdots$$

$$T_k = Y_k - Y_{k-1}$$

T_k is geometric(P)

can also define

$$N_t = X_1 + X_2 + \cdots + X_t \quad (t = 1, 2, \ldots)$$

$$= \#\text{ arrivals in } [1, t]$$

N_t is Binomial(t, p). We can recover X_t from seq. $N_1, N_2, \ldots$

$$X_1 = N_1$$

$$X_t = N_t - N_{t-1} \quad t = 2, 3, \ldots$$

Can also recover

$$Y_k = \min\{t \mid N_t = k\} \quad k = 1, 2, \ldots$$

Consider an arrival Process that evolves in continuous time

<u>Define</u>:

$$P(k, \tau) = P\left(\begin{array}{l}\text{\# of arrivals during} \\ \text{an interval of length} \\ \tau \text{ is } k\end{array}\right)$$

This Process is called a <u>Poisson Process</u> iff

(a) <u>time homogeniety</u>

$P(k, \tau)$ is same for all intervals of length τ.

Analogous to Prob. of success is p for all trials.

(b) <u>Independence</u>

the # of arrivals in disjoint time intervals are independent.

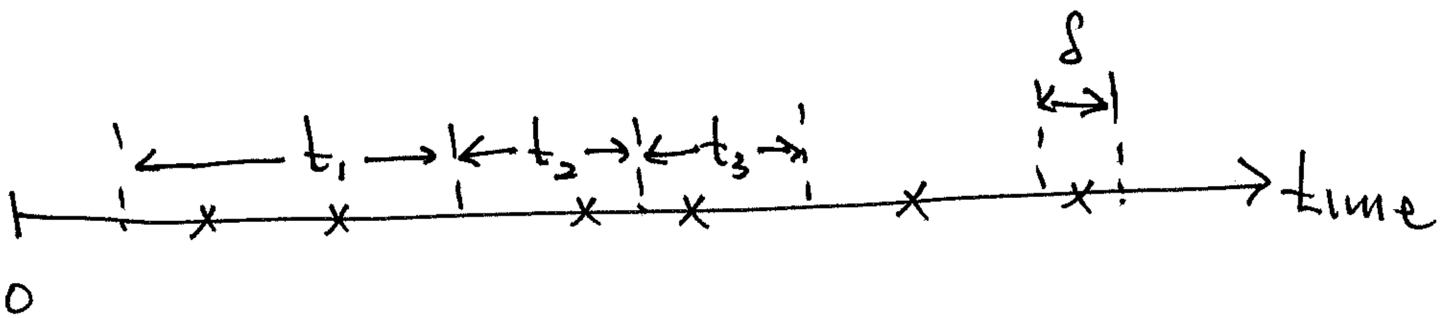
Analogous to different trials being independent in Bernoulli.

(c) <u>Small interval Probabilities</u>

Let $\delta > 0$ be small. Then

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k=2,3,\ldots \end{cases}$$

Here λ is the <u>arrival rate</u>, or <u>intensity</u>.



$$P(k,\delta) = \left\{ \begin{array}{ll} 1-\lambda\delta & k=0 \\ \lambda\delta & k=1 \\ 0 & k=2,3,\dots \end{array} \right\} + o(\delta)$$

where $o(\delta)$ denotes a function with the property

$$\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$$

For instance e.g. δ^2 has this property.

Observe

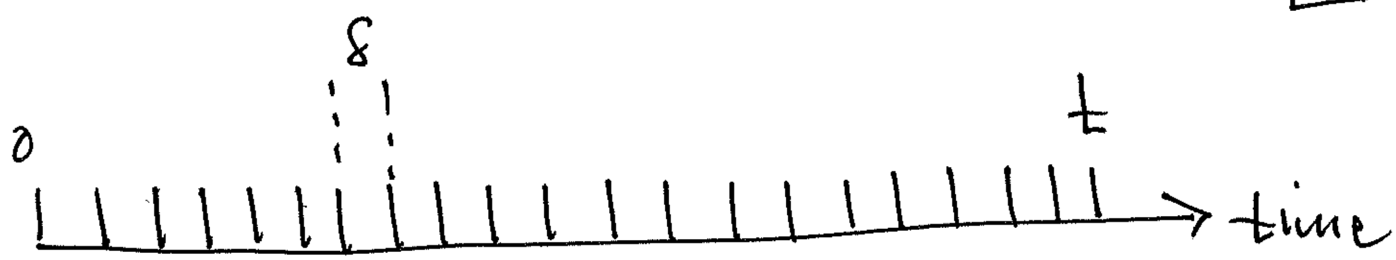
$$E[\#\text{arrivals in } [t, t+\delta]]$$

$$= (1-\lambda\delta)\cdot 0 + \lambda\delta\cdot 1 + 0\cdot(\dots)$$

$$= \lambda\delta$$

$\therefore$ λ is expected # arrivals per unit time.

What about # arrivals in a large interval $[0, t]$



let $n = \frac{t}{\delta}$ be # of subintervals

choose δ so small (i.e. n large) that $P(k,\delta) \approx 0$ for $k \geq 2$, i.e. Prob. of having more than 1 arrival in negligable.

we have (approximately) a Bernoulli Process with Parameter

$$p = \lambda \delta = \frac{\lambda t}{n}$$

approximation improves as $\delta \to 0$ (i.e. $n \to \infty$).

The # of arrivak in n trials of a Bernoulli Process is Binomial $(n, p|$ so

approx.

$$P(k \text{ arrivals in } [0,t])$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \cdot \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

In limit as $n \to \infty$ $(\delta \to 0)$ we have Poisson distribution.