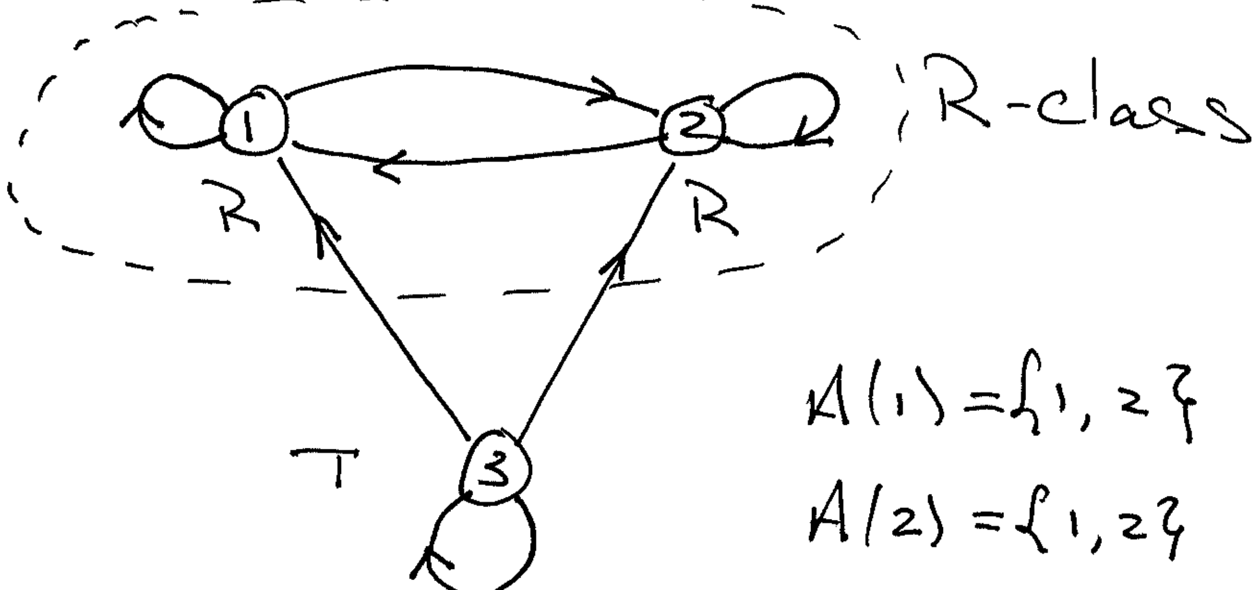
cse 107 3-14-24

Note:

Transience & recurrence are determined by the arcs in the state-transition diagram, not by the actual probabilities P_{ij} on each arc.

Ex.

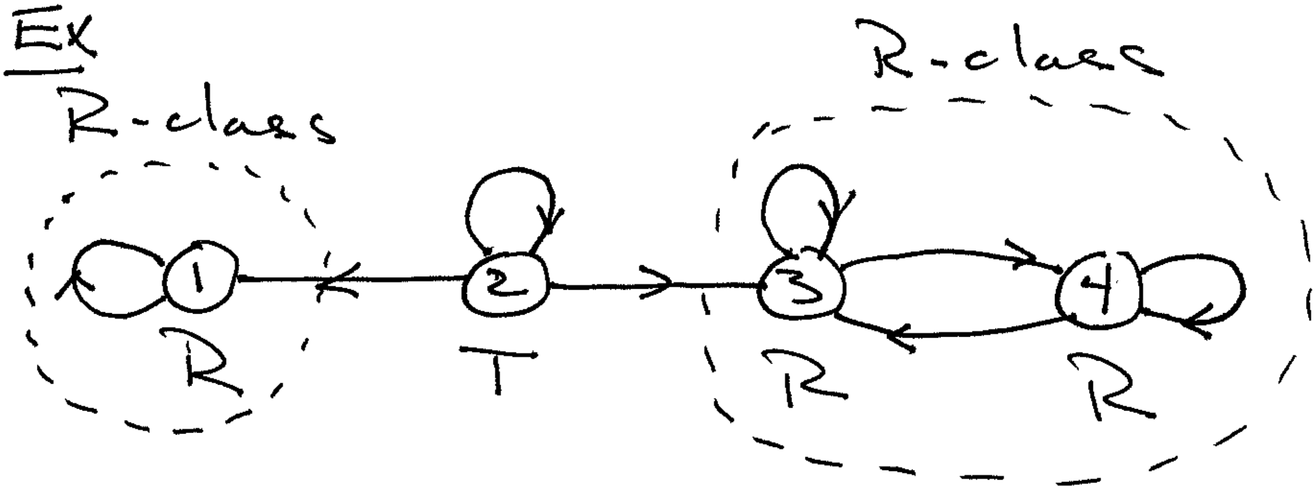


$A(1) = \{1, 2\}$

$A(2) = \{1, 2\}$

$A(3) = \{1, 2, 3\}$

Ex



$A(1) = \{1\}$

$A(2) = \{1, 2, 3, 4\}$

$A(3) = \{3, 4\}$

$A(4) = \{3, 4\}$

note that for any states i, j

$$j \in A(i) \Rightarrow A(j) \subseteq A(i)$$

and

$$i \in A(j) \Rightarrow A(i) \subseteq A(j)$$

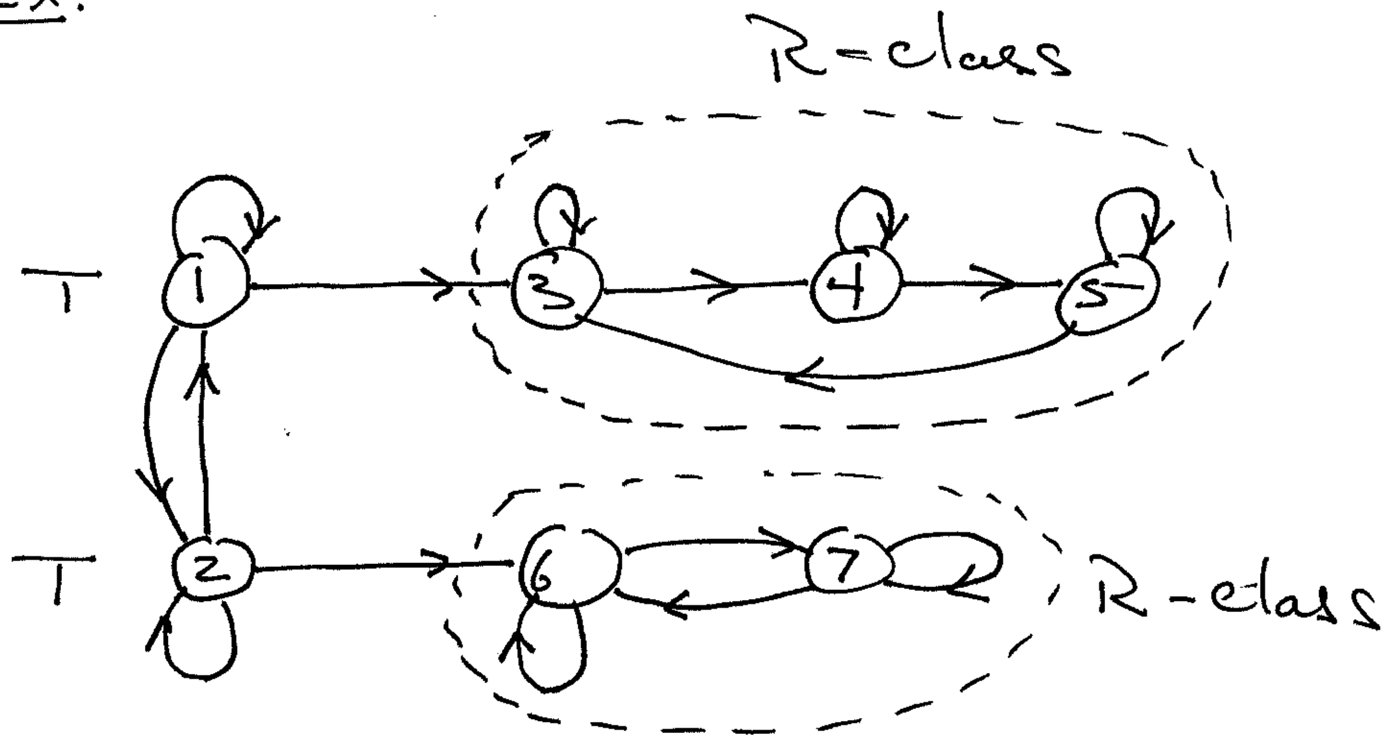
Therefore, if j is recurrent, then

$$\left.\begin{array}{c} j \in A(i) \Rightarrow A(j) \subseteq A(i) \\ \Updownarrow \\ i \in A(j) \Rightarrow A(i) \subseteq A(j) \end{array}\right\} \Rightarrow A(i) = A(j)$$

<u>Defn</u>

If i is recurrent, the set $A(i) \subseteq S$ is called a <u>recurrent class</u>

Ex.



$A(1) = A(2) = S = \{1, 2, 3, 4, 5, 6, 7\}$

$A(3) = A(4) = A(5) = \{3, 4, 5\}$

$A(6) = A(7) = \{6, 7\}$

Exercise (see #6 on P. 381)

Let i be a transient state.
Prove there exists a recurrent state j such that

$$i \dashrightarrow j$$

i.e. $\quad j \in A(i)$

Theorem (Markov chain Decomposition)

(1) A Markov chain can be decomposed into one or more recurrent classes and possibly some transient states.

(2) A recurrent state is accessible from all states in its class, but not from ~~other~~ states in other recurrent classes.

(3) A transient state is not accessible from any recurrent state.

(4) At least one recurrent state is accessible from a given transient state.

Facts

- when a Markov Chain enters a recurrent class, it stays in that class, and all states in that class will be visited infinitely often.

- If the initial state is transient, then its trajectory consists of a finite sequence of transient states, followed by an infinite sequence of recurrent states, all in one class.

Defn

A recurrent class is called Periodic if its states can be grouped into subsets

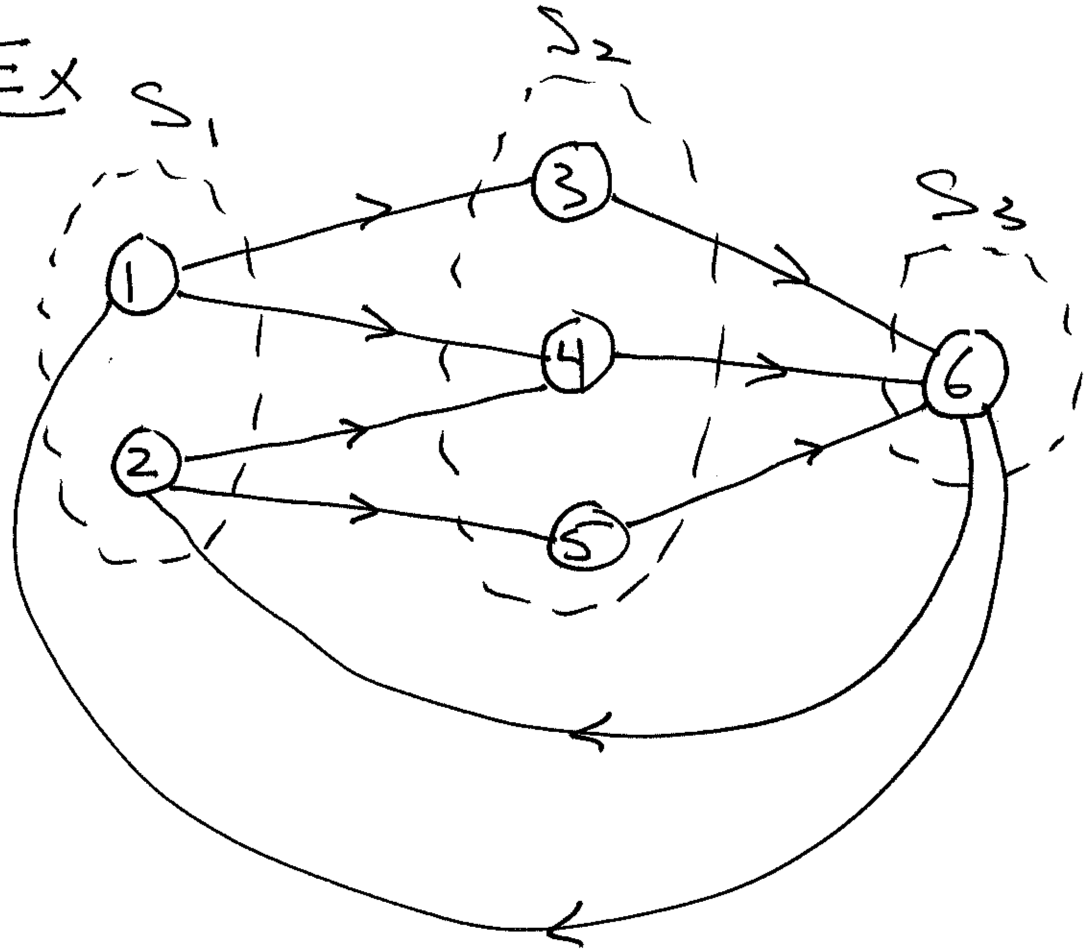
$$S_1, S_2, \ldots, S_d \quad (d \geq 2)$$

such that all transitions from S_k lead to S_{k+1} $(1 \leq k < d)$ and S_d lead to S_1.

$$\begin{tikzcd} S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \cdots \arrow[r] & S_{d-1} \arrow[r] & S_d \arrow[lllll, bend left] \end{tikzcd}$$

d is called the Period.

Ex



$d=3$

observe: if $X_0 \in S_1$, then

$$X_n \in S_1 \text{ iff } n \equiv 0 \pmod 3$$

$$X_n \in S_2 \text{ iff } n \equiv 1 \pmod 3$$

$$X_n \in S_3 \text{ iff } n \equiv 2 \pmod 3$$

Defn

A recurrent class C is aperiodic iff it is not periodic. Equivalently, there exists a time $n \geq 0$ such that

$$r_{ij}(n) > 0 \quad \text{for all } i,j \in C$$

Ex. Add a self loop to any state in last example.

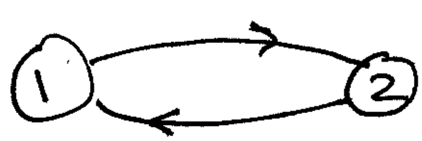
7.3 Steady state behavior

what spoils limiting behavior ?

- multiple recurrent classes

or

- Periodic recurrent class



we write

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j \quad \text{(for all } i\text{)}$$

when the limit exists, and we call π_j the <u>steady-state probability</u> of j.

Interpretation

$$\pi_j \approx P(X_n = j) \quad \text{(large } n\text{)}$$

<u>Theorem</u> (steady state convergence)

consider a Markov chain with a single recurrent class, which is aperiodic. Then for each $j \in S$:

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j \quad (\text{all } i)$$

exist. Further, π_j are solutions to

$$(1) \qquad \sum_{k=1}^{m} \pi_k P_{kj} = \pi_j \quad (j = 1, 2, .., m)$$

$$(2) \qquad \sum_{k=1}^{m} \pi_k = 1$$

Also $\pi_j = 0$ if j is transient, and $\pi_j > 0$ if j is recurrent.

(1) are balance eqns & (2) is the normalization eqn.

(1) follows from C-K eqns!

$$\underbrace{r_{ij}(n)}_{\downarrow \atop \pi_j} = \sum_{k=1}^{m} \underbrace{r_{ik}(n-1)}_{\downarrow \atop \pi_k} p_{kj}$$

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}$$

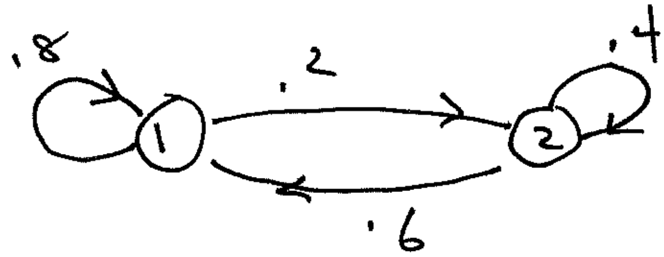
<u>matrix</u>

$$R^n = R^{n-1} \cdot R$$

$$\downarrow \qquad \downarrow$$

$$\pi = \pi \cdot R$$

Ex.



$$(1)\quad \begin{cases} \pi_1 P_{11} + \pi_2 P_{21} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} = \pi_2 \end{cases}$$

$$(2)\quad \{\, \pi_1 + \pi_2 = 1$$

$$\begin{cases} .8\pi_1 + .6\pi_2 = \pi_1 \rightarrow 6\pi_2 = 2\pi_1 \\ .2\pi_1 + .4\pi_2 = \pi_2 \rightarrow 2\pi_1 = 6\pi_2 \end{cases}$$

$$\rightarrow \quad \pi_1 = 3\pi_2$$

$$\pi_1 + \pi_2 = 1 \rightarrow 3\pi_2 + \pi_2 = 1$$

$$\therefore \quad 4\pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}}$$

$$\therefore \quad \boxed{\pi_1 = \frac{3}{4}}$$

Equations (1) are called the <u>balance equations</u>, and follow directly from the Chapman-Kolmogorov equations

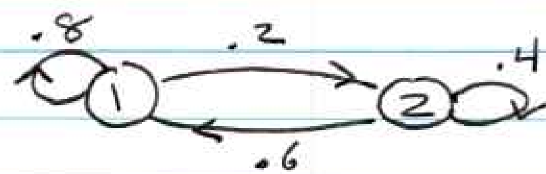
$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$

By taking limits $r_{ij}(n) \to \pi_j$ and $r_{ik}(n-1) \to \pi_k$ as $n \to \infty$.

Equations (2) are called the <u>normalization equations</u>, and follow from the fact that

$$\sum_{j=1}^{m} P(X_n = j) = 1 \quad (\text{any } n \geq 0)$$

Ex



$$p_{11} = .8, \quad p_{12} = .2$$
$$p_{21} = .6, \quad p_{22} = .4$$

$$(1) \quad \begin{cases} \pi_1 p_{11} + \pi_2 p_{21} = \pi_1 \\ \pi_1 p_{12} + \pi_2 p_{22} = \pi_2 \end{cases}$$

$$(2) \quad \pi_1 + \pi_2 = 1$$

$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 = \pi_1 \leftrightarrow (.6)\pi_2 = (.2)\pi_1 \\ (.2)\pi_1 + (.4)\pi_2 = \pi_2 \leftrightarrow (.2)\pi_1 = (.6)\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

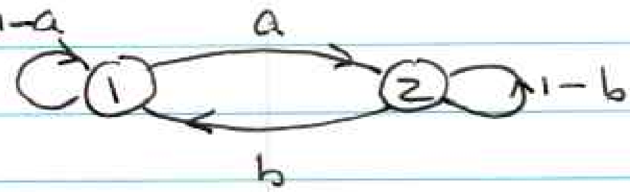
The 1st two give $\pi_1 = 3\pi_2$. The 3rd yields

$$3\pi_2 + \pi_2 = 1 \rightarrow 4\pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}}$$

hence

$$\boxed{\pi_1 = \frac{3}{4}}$$

Ex.



$$\pi_1(1-a) + \pi_2 b = \pi_1$$
$$\pi_1 a + \pi_2(1-b) = \pi_2$$

$$\pi_2 b = \pi_1 a$$

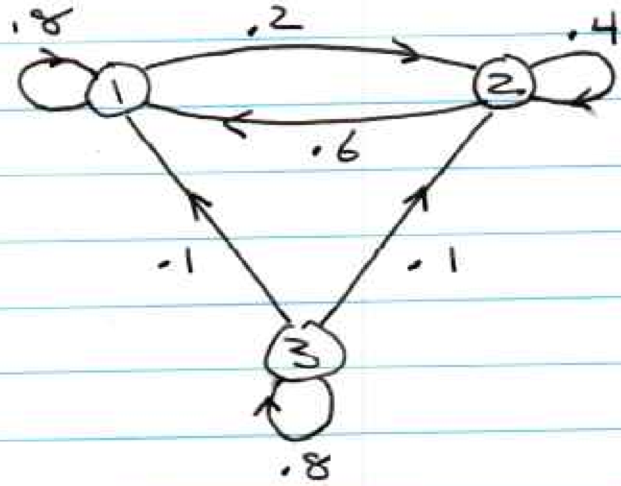
$$\therefore \quad \pi_1 = \frac{b}{a}\pi_2$$

$$\therefore \quad \frac{b}{a}\pi_2 + \pi_2 = 1$$

$$\therefore \quad \pi_2 = \frac{1}{\frac{b}{a}+1} \rightarrow \boxed{\pi_2 = \frac{a}{a+b}}$$

$$\therefore \quad \boxed{\pi_1 = \frac{b}{a+b}}$$

Ex



$$\begin{cases} \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} = \pi_2 \\ \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} = \pi_3 \end{cases}$$

$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 + (.1)\pi_3 = \pi_1 \\ (.2)\pi_1 + (.4)\pi_2 + (.1)\pi_3 = \pi_2 \end{cases}$$

$$\begin{cases} -2\pi_1 + 6\pi_2 + \pi_3 = 0 \\ 2\pi_1 - 6\pi_2 + \pi_3 = 0 \end{cases} \rightarrow 2\pi_3 = 0 \rightarrow \boxed{\pi_3 = 0}$$

$$\pi_1 = 3\pi_2$$

$$3\pi_2 + \pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}} \therefore \boxed{\pi_1 = \frac{3}{4}}$$

See examples 7.6 and 7.7 on pages 355 and 356, resp.

we should have anticipated this since we know state 3 is transient