CSE 107 3-12-24

- SETs : Due Sunday 3/17 11:59 PM
- Final Exam: Wed. 3/20 4:00-5:30 PM

if Response ≥ 85%
I will add .5%
to all overall scores

Exercise

Define a Probability matrix to be a square ($n \times n$) matrix in which each row sums to 1. i.e. $A = (a_{ij})$ and

$$\sum_{j=1}^{n} a_{ij} = 1 \quad (\text{for all } 1 \le i \le n).$$

Prove that the product of probability matrices is another probability matrix.

Note: if $A \cdot B = C$ then

$$C = (c_{ij})$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Chapman Kolmogorov Eqns

The n-step trans. Probabilities $r_{ij}(n)$ satisfy

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj} \quad (i,j \in S, \; n \geq 2)$$

Note:

recall : $P(A|B|C) = P(A \mid B \cap C)$.

The total Prob. thm.

$$P(A|C) = \sum_{k=1}^{m} P(A|B_i \cap C) \cdot P(B_i|C)$$

Proof (of Chap.-Kol.)

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

$$= \sum_{k=1}^{m} P(X_n = j \mid X_{n-1} = k, X_0 = i) \cdot P(X_{n-1} = k \mid X_0 = i)$$

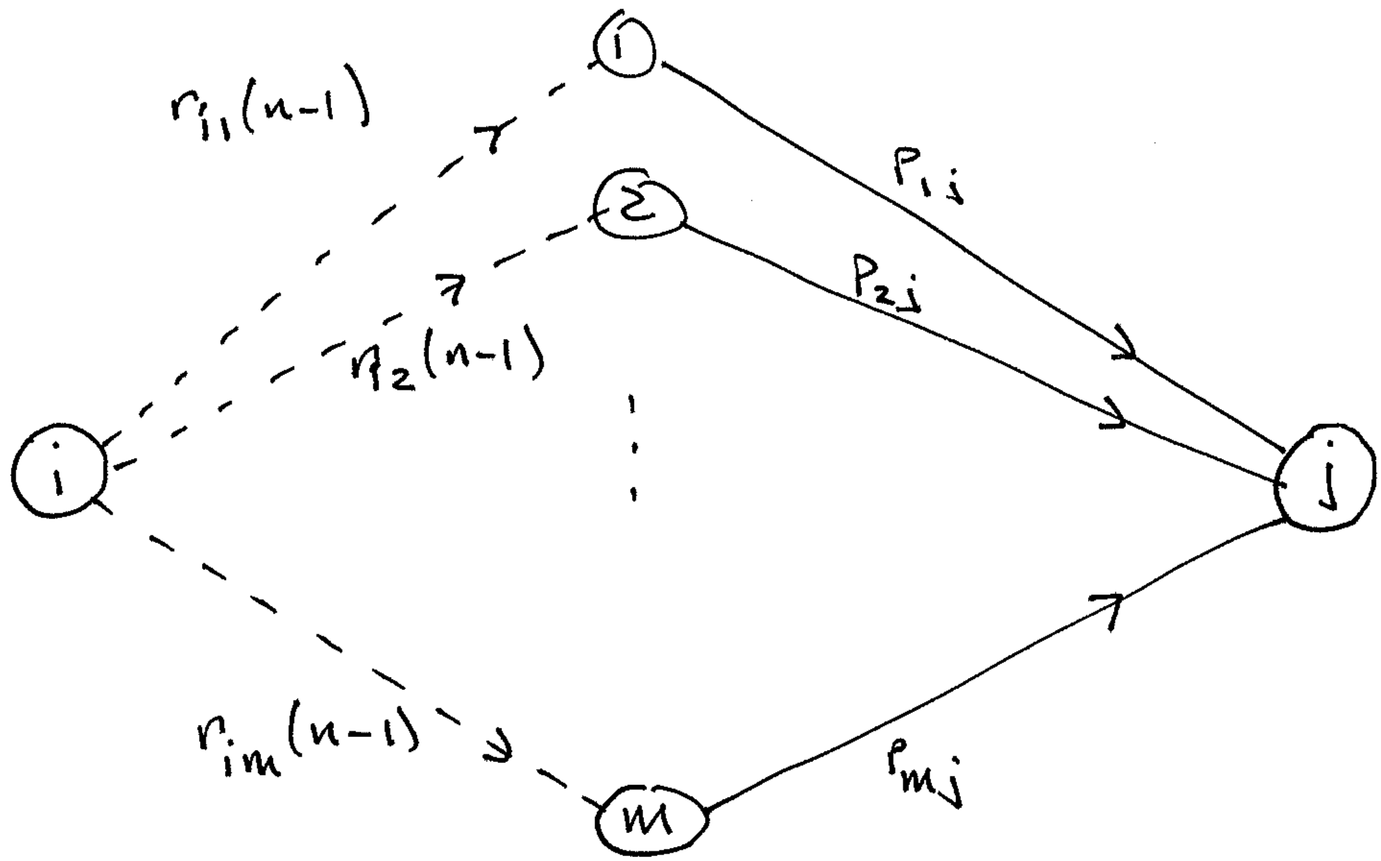
$$= \sum_{k=1}^{m} P(X_{n-1} = k \mid X_0 = i) \cdot \underbrace{P(X_n = j \mid X_{n-1} = k)}$$

by Markov Property

$$= \sum_{k=1}^{m} r_{ik}(n-1) \cdot P_{kj}$$

■

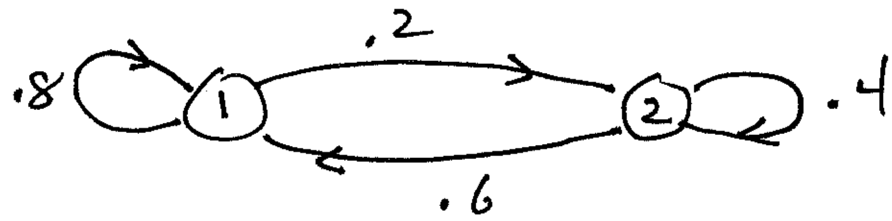
Picture



CK Eqns ! $$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj}$$

Matrix Form } $$\left(r_{ij}(n)\right) = R^{n-1} \cdot R = R^{n}$$

Ex. Previous



$$R = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

we have

$$R^3 = \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix} \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

$$= \begin{pmatrix} .752 & .248 \\ .744 & .256 \end{pmatrix}$$

so for instance

$$P(X_3 = 1 \mid X_0 = 2) = .744 = r_{21}(3)$$

with some effort

$$R^{10} = \begin{pmatrix} .7500000256 & .2499999744 \\ .7499999232 & .2500000768 \end{pmatrix}$$

so $P(X_{10}=1 \mid X_0=2) = r_{21}(10) = .7499999232$.

It can be shown

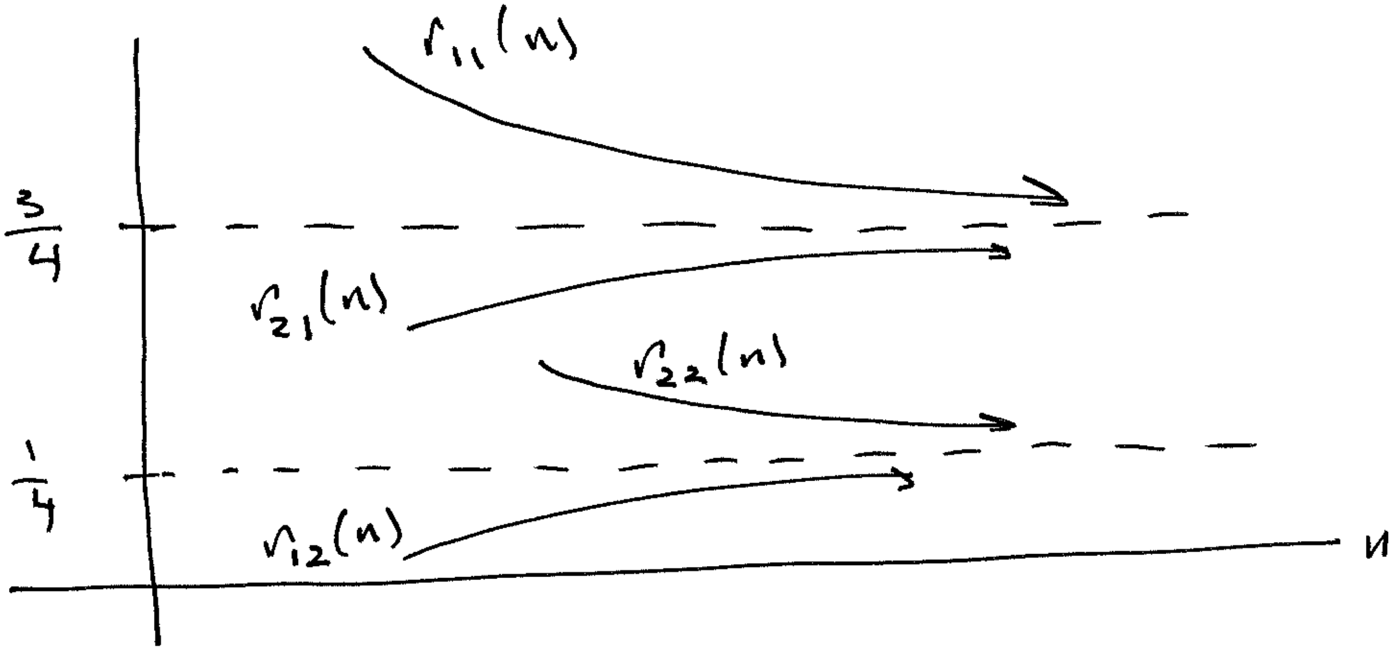
$$\lim_{n\to\infty} R^n = \begin{pmatrix} .75 & .25 \\ .75 & .25 \end{pmatrix}$$

i.e.

$$r_{11}(n) \longrightarrow \tfrac{3}{4}^{+}, \quad r_{12}(n) \longrightarrow \tfrac{1}{4}^{-}$$

$$r_{21}(n) \longrightarrow \tfrac{3}{4}^{-}, \quad r_{22}(n) \longrightarrow \tfrac{1}{4}^{+}$$

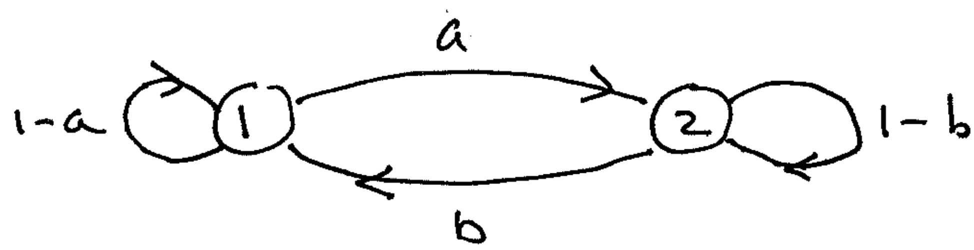
as $n \to \infty$.



Thus, if we wait a long time

$$\underbrace{P(X_n = 1) \approx \frac{3}{4} \text{ and } P(X_n = 2) \approx \frac{1}{4}}_{\text{steady state Probabilities}}$$

Ex.



where $0 < a < 1$, $0 < b < 1$.

$$R = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

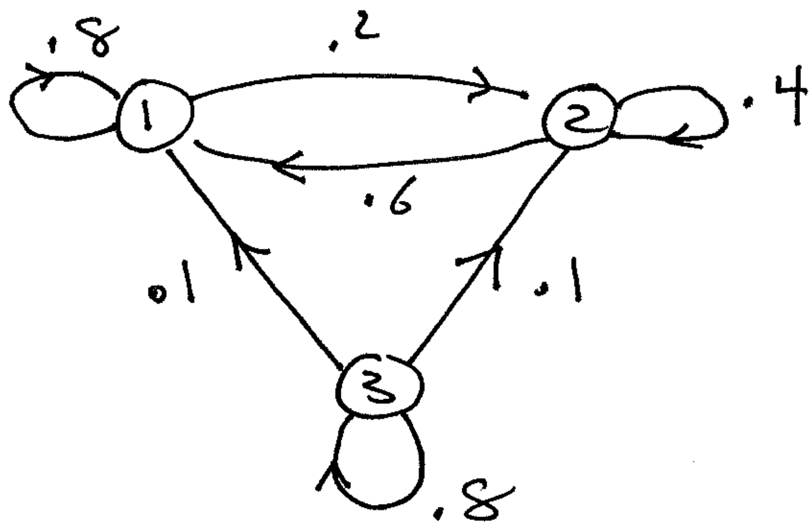
one can show

$$\lim_{n\to\infty} R^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

so $$\lim_{n\to\infty} P(X_n = 1) = \frac{b}{a+b}$$

$$\lim_{n\to\infty} P(X_n = 2) = \frac{a}{a+b}$$

Ex.



$$R = \begin{pmatrix} .8 & .2 & 0 \\ .6 & .4 & 0 \\ .1 & .1 & .8 \end{pmatrix}$$

we can show

$$\lim_{n\to\infty} R^n = \begin{pmatrix} .75 & .25 & 0 \\ .75 & .25 & 0 \\ .75 & .25 & 0 \end{pmatrix}$$

7.2 Classification of states

Defn

We say state j is accessible from state i (or that j is reachable from i) iff

$$r_{ij}(n) > 0 \quad \text{for some } n \geq 1$$

Equivalently, there exists a seq. of states $i, i_1, i_2, \ldots, i_{n-1}, j$ such that the transitions

$$i \to i_1 \to i_2 \to \cdots \to i_{n-1} \to j$$

all have positive probability

<u>notation</u>: $i \dashrightarrow j$

<u>Defn</u>

Let $A(i)$ denote the set of all states accessible from i

i.e.

$$A(i) = \{ j \in S \mid i \dashrightarrow j \}$$

<u>Defn</u>

A state i is called <u>recurrent</u> if for every j accessible from i, we also have i accessible from j.

Equivalently

$$j \in A(i) \iff i \in A(j)$$

i.e.

$$\textcircled{i} \dashrightarrow \textcircled{j} \iff \textcircled{j} \dashrightarrow \textcircled{i}$$

Thus, if we depart from a recurrent recurrent state, there is a positive probability of returning, so if we wait long enough, we will return.

Hence a recurrent state, if it is ever visited, will be visited infinitely often.

Defn

A state i is called <u>transient</u> iff it is not recurrent. Thus i is transient iff there exists a state j with

$$j \in A(i) \quad \text{and} \quad i \notin A(j)$$

Thus after a visit to a transient state i, there is a positive probability of entering such a state j. Given enough time this is certain to happen.

Hence a transient state will only be visited finitely many times

Ex.