CSE 107 2-9-24

Supplemental Lecture

---

<u>Goal</u>: Compare CDFs of Geometric & exponential r.v.s.

Let $X$ be geometric with Param. $p$.

<u>PMF</u>: $P_X(k) = p(1-p)^{k-1} \quad (k=1,2,3,\dots)$

<u>CDF</u>: $F_{geo}(n) = \sum_{k=1}^{n} p(1-p)^{k-1}$

$$= p \cdot \sum_{k=1}^{n} (1-p)^{k-1}$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}} = \boxed{1-(1-p)^n}$$

$n = 1, 2, \dots$

another way:

$$F_{geo}(n) = P(X \le n) = 1 - P(X > n) = 1-(1-p)^n.$$

Let $Y$ be exponential with Param. $\lambda$.

Then

$$f_Y(x) = \lambda e^{-\lambda x} \qquad (x \geq 0)$$

Thus, for $x \geq 0$, we have

$$F_{exp}(x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t}\Big|_0^x = -\left(e^{-\lambda x} - 1\right)$$

$$= \boxed{1 - e^{-\lambda x}}$$

To compare $F_{exp.}$ and $F_{geo}$, pick $\delta > 0$ so that

$$e^{-\lambda\delta} = 1 - p \qquad (\text{i.e. } p = 1 - e^{-\lambda\delta})$$

i.e. $$-\lambda\delta = \ln(1-p)$$

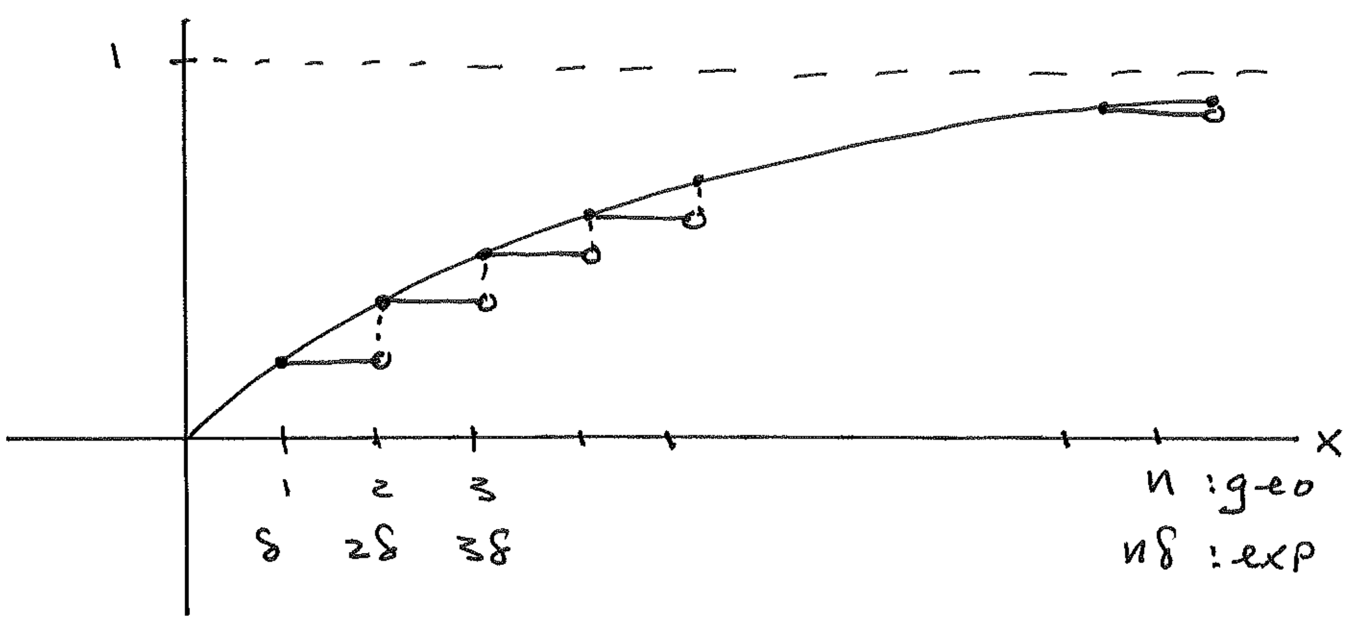
i.e. $$\delta = \frac{-\ln(1-p)}{\lambda}$$

For this value of $\delta$, we have

$$\left(e^{-\lambda\delta}\right)^n = (1-p)^n$$

$$\therefore \quad 1-e^{-\lambda\delta n} = 1-(1-p)^n$$

$$\therefore \quad F_{exp}(\delta n) = F_{geo}(n)$$



Thus if we toss quickly (every $\delta$ seconds) with probability of heads

$$p = 1 - e^{-\lambda\delta}$$

then the 1st time "n" to get heads is an approx. to the exponential at time $\delta n$.

## 3.3 Normal Random Variables

Let $X$ be a cont. r.v. with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

we call $X$ <u>normal</u> or <u>Gaussian</u> random variable. we can show

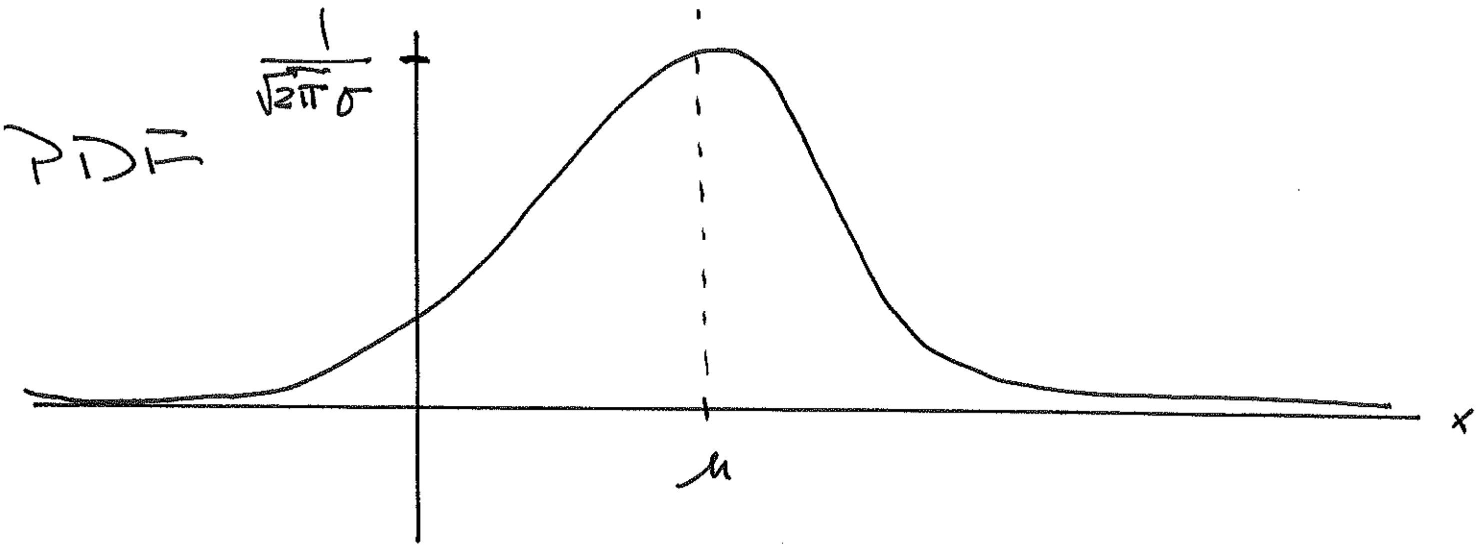
$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

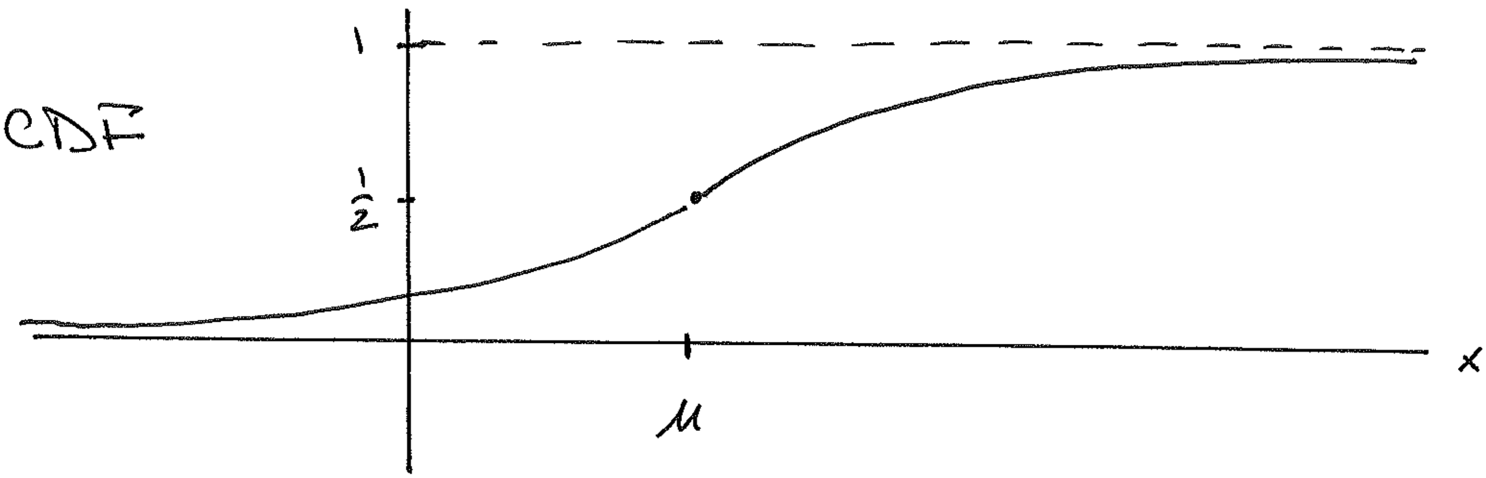
we can also show

$$E[X] = \mu$$

$$Var(X) = \sigma^2$$

$$\sqrt{Var(X)} = \sigma$$




It is evident that: $f_X(-(x-\mu)) = f_X(x-\mu)$

$$x-\mu \longleftrightarrow -(x-\mu) = \mu - x$$

hence $f_X$ is symmetric about $x=\mu$.

$\therefore\ E[X] = \mu$.

**Theorem**

If $X$ is any cont. r.v. with PDF satisfying $f_X(x) = f_X(-x)$ $\forall x \in \mathbb{R}$, then $E[X] = 0$

**Proof**: exercise

**Theorem**

If $X$ is a normal r.v. with param. $\mu$ & $\sigma^2$, and $a, b \in \mathbb{R}, a \neq 0$, then

$$Y = aX + b$$

is also normal with

$$E[Y] = a\mu + b$$
$$Var(Y) = a^2\sigma^2$$

Note:

we already knew formulas for $E[Y]$ and $Var(Y)$. What's new is the fact that $Y$ is normal.

Proof later

A normal r.v. $Y$ is called standard normal iff its mean is $0$ and its variance is $1$.

PDF: $f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$

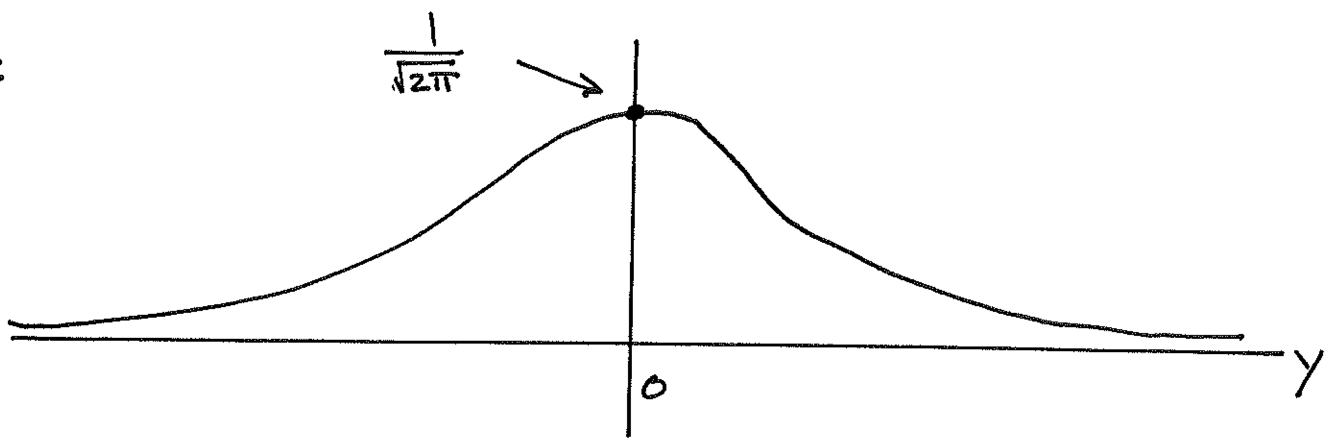
The CDF of the standard normal is denoted $\Phi(y)$.

so

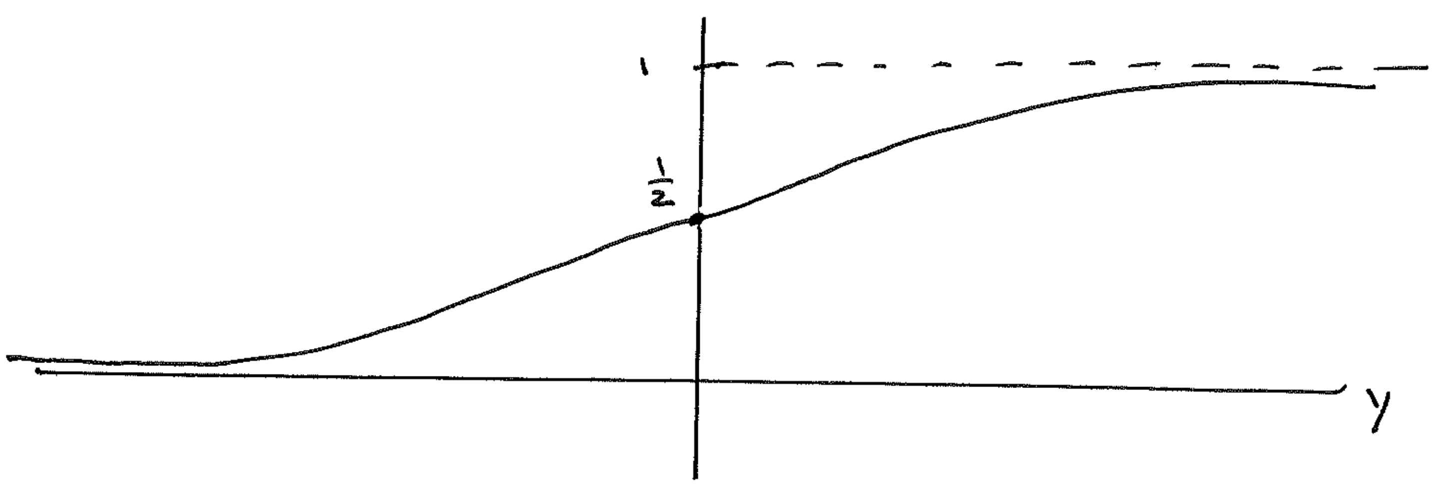
$$\Phi(y) = P(Y \le y) = P(Y < y)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\, dt$$

PDF




CDF



Handout Standard Normal Table gives values of $\Phi(y)$ for

$$0 \leq y \leq 3.49$$

If $y > 3.49$, then $\Phi(y) \approx 1$. The identity

$$\Phi(-y) = 1 - \Phi(y) \qquad (y \in \mathbb{R})$$

Allows us to compute $\Phi(y)$ for $y < 0$

<u>Exercise</u>: Prove $\Phi(-y) = 1 - \Phi(y)$ for all $y \in \mathbb{R}$. Hint: use symmetry of $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$.

Ex.

- $\Phi(1.84) = .9671$
- $\Phi(-1.84) = 1 - \Phi(1.84)$

$$= 1 - .9671$$

$$= .0329$$

Suppose $X$ is normal with mean, and variance $\mu, \sigma^2$, resp. How do we compute values of $F_X(x)$?

Let $Y = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$.

Then $Y$ is normal with

$$E[Y] = \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$$

and $Var(Y) = \frac{1}{\sigma^2}Var(X) = \frac{\sigma^2}{\sigma^2} = 1$.

Hence $Y$ is std. normal. Thus

$$F_X(x) = P(X \le x)$$

$$= P(X - \mu \le x - \mu)$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Y \le \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right).$$