CSE 107 2-8-24

Exercise: let X be exponential r.v.

(a) show $E[X] = \frac{1}{\lambda}$ ✓

(b) show $Var(X) = \frac{1}{\lambda^2}$

Proof of (a):

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = -\int_0^{\infty} x\left(-\lambda e^{-\lambda x}\right) dx$$

integration by parts
$$\begin{cases} u = x & dv = -\lambda e^{-\lambda x} dx \\ du = dx & v = e^{-\lambda x} \end{cases}$$

$$= -\left(x e^{-\lambda x} \Big|_0^\infty - \int_0^\infty e^{-\lambda x}\, dx \right)$$

$$= -\left((0-0) - \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^\infty \right)$$

$$= -\left(\frac{1}{\lambda}(0-1) \right) = \boxed{\frac{1}{\lambda}} \quad \checkmark$$

<u>note:</u>

- X has units: $\dfrac{\text{time}}{\text{arrival}}$
- $E[X]$ " " $\dfrac{\text{time}}{\text{arrival}}$
- $\dfrac{1}{\lambda}$ " " "
- λ " " $\dfrac{\text{arrival}}{\text{time}}$

so λ is called the <u>arrival rate</u>.

If X is exponential (Param. λ), then

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

<u>Proof</u>

let $a > 0$. Then

$$P(X \geq a) = -\int_a^\infty -\lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_a^\infty = -\left(0 - e^{-\lambda a}\right)$$

$$= e^{-\lambda a}.$$

Ex.

The time X until next earthquake (4.0 or above) is modeled as an exponential r.v. with mean 53 minutes (worldwide). What the Prob. that such an earthquake occurs in the next 10 minutes.

Solution

$$E[X] = \frac{1}{\lambda} = 53 \Rightarrow \boxed{\lambda = \frac{1}{53}}.$$

We seek

$$P(0 \le X \le 10) = P(X \ge 0) - P(X > 10)$$

$$= 1 - e^{-\frac{10}{53}}$$

$$= \boxed{.1719}$$

<u>Exercise</u>

determine $a \in \mathbb{R}$ such that the Prob. of an earthquake within the next a minutes is $.99$, $a > 0$

i.e.

$$P(0 \le X \le a) = .99$$

3.2 Cumulative Distribution Functions

Defn

let X be a r.v.. The cumulative distribution function (CDF) of X is

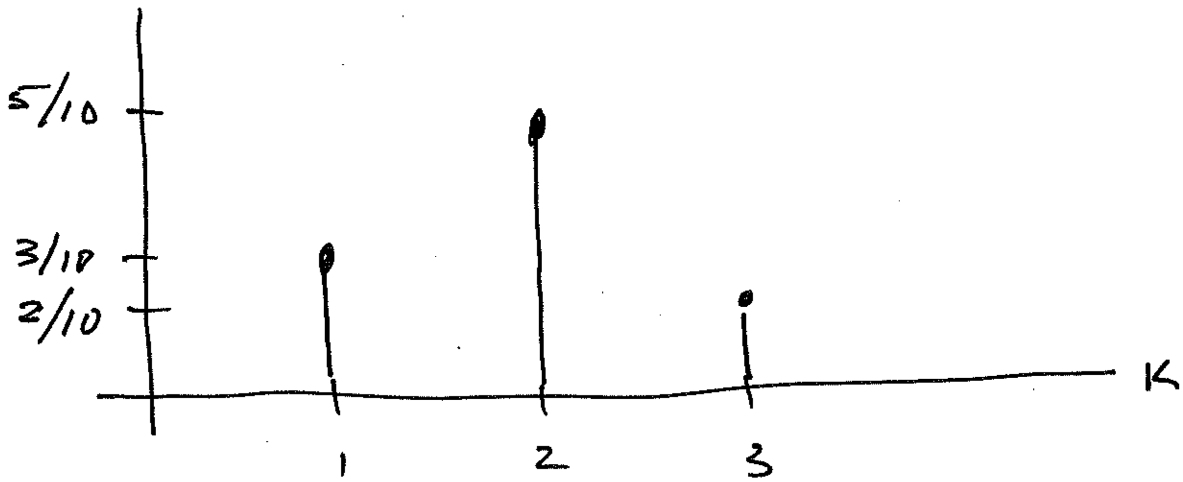
$$F_X(x) = P(X \le x)$$

note

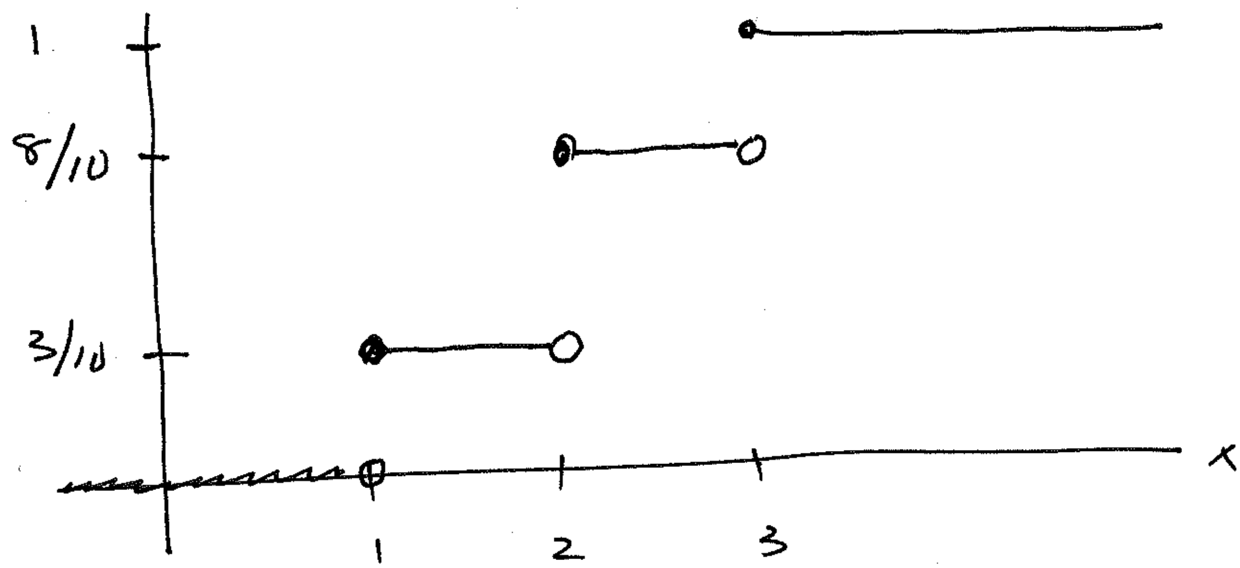
$$F_X(x) = \begin{cases} \displaystyle\sum_{k \le x} P_X(k) & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{x} f_X(t)\,dt & \text{if } X \text{ is continuous} \end{cases}$$

Ex (Discrete)

PMF: $$P_X(k) = \begin{cases} 3/10 & \text{if } k=1 \\ 5/10 & k=2 \\ 2/10 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

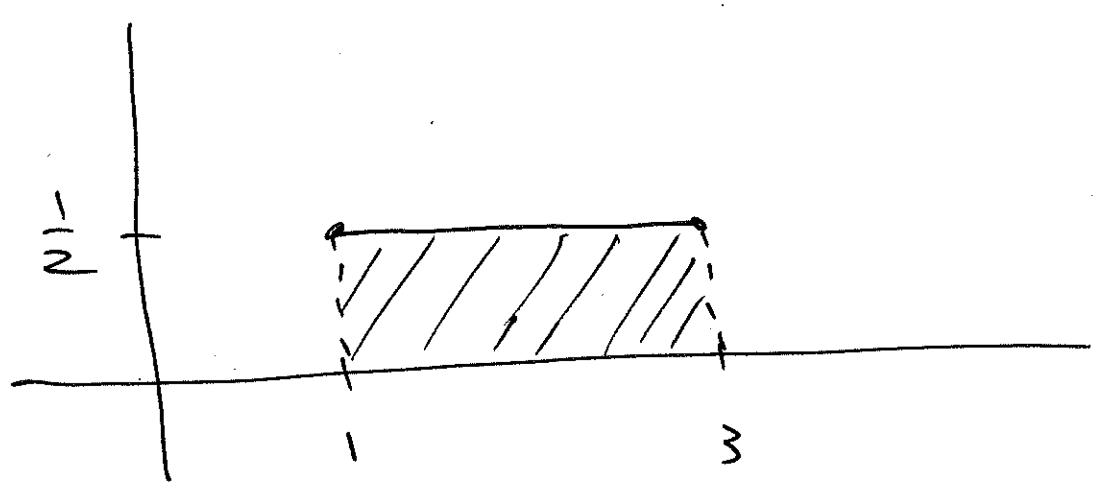


CDF:
$$F_X(x) = \begin{cases} 0 & \text{if } x<1 \\ 3/10 & 1 \le x < 2 \\ 8/10 & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$



<u>Ex</u> (Continuous) uniform on [1, 3]

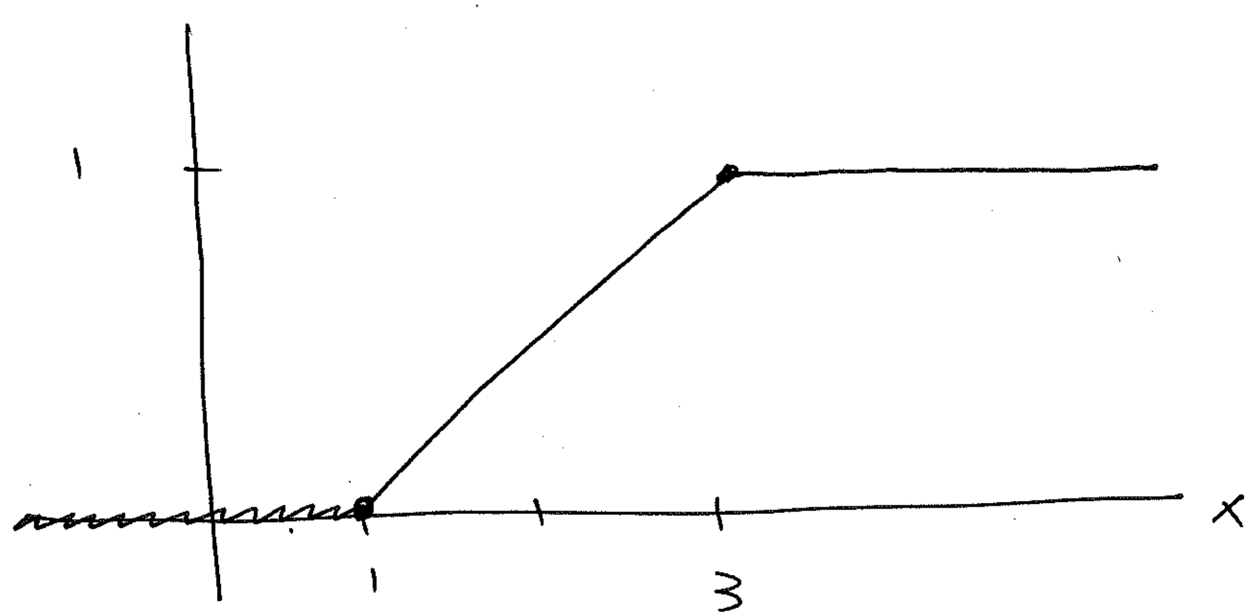
$$\text{PDF}: \ f_X(x) = \begin{cases} 1/2 & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$



for x in range $1 \le x \le 3$:

$$F_X(x) = \int_1^x \tfrac{1}{2}\,dt = \tfrac{1}{2}t\Big|_1^x = \tfrac{1}{2}(x-1)$$

CDF : $$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x-1}{2} & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$



- note: the CDF

$$F_X(x) = P(X \le x)$$

is always monotonically non-decreasing.

i.e. if $x \le y$, then $F_X(x) \le F_X(y)$

- If X is discrete, then $F_X(x)$ is piecewise constant

$$F_X(k) = \sum_{i \le k} p_X(i)$$

and

$$\begin{aligned} p_X(k) &= P(X = k) \\ &= P(X \le k) - P(X \le k-1) \\ &= F_X(k) - F_X(k-1) \end{aligned}$$

(assuming X takes integer values)

- If X is continuous, then $F_X(x)$ is a continuous function. Also

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

and

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Ex.

You take a test 3 times. Your final score X is the maximum of the three scores:

$$X = \max(X_1, X_2, X_3)$$

where X_i is your score on i'th test.

Assume X_i are independent r.v.s, and uniform over $\{1, 2, 3, \dots, 10\}$.

Thus

$$P_{X_i}(k) = \begin{cases} 1/10 & \text{if } k = 1, 2, \dots, 10 \\ 0 & \text{otherwise} \end{cases}$$

Find the PMF of X.

Solution

First find CDF $F_X(k)$, then difference to get

$$P_X(k) = F_X(k) - F_X(k-1).$$

so

$$F_X(k) = P(X \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k)$$

by indep.

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \left(\frac{k}{10}\right) \cdot \left(\frac{k}{10}\right) \cdot \left(\frac{k}{10}\right)$$

more precisely:

$$F_{X_i}(k) = \begin{cases} 0 & \text{if } k < 1 \\ \frac{k}{10} & \text{if } k = 1, 2, \ldots, 10 \\ 1 & \text{if } k > 10 \end{cases}$$

Thus

$$F_X(k) = F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \begin{cases} 0 & \text{if } k < 1 \\ \left(\frac{k}{10}\right)^3 & \text{if } k = 1, 2, \ldots, 10 \\ 1 & \text{if } k > 10. \end{cases}$$

And

$$p_X(k) = F_X(k) - F_X(k-1)$$

$$= \begin{Bmatrix} 0 & K<1 \\ \left(\frac{K}{10}\right)^3 & 1 \le K \le 10 \\ 1 & K > 10 \end{Bmatrix} - \begin{Bmatrix} 0 & K-1<1 \\ \left(\frac{K-1}{10}\right)^3 & 1 \le K-1 \le 10 \\ 1 & K-1 > 10 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & K<1 \\ \left(\frac{K}{10}\right)^3 & 1 \le K \le 10 \\ 1 & K > 10 \end{Bmatrix} - \begin{Bmatrix} 0 & K<2 \\ \left(\frac{K-1}{10}\right)^3 & 2 \le K \le 11 \\ 1 & K > 11 \end{Bmatrix}$$

$$= \begin{cases} 0 & K<1 \\ \left(\frac{1}{10}\right)^3 & K=1 \\ \left(\frac{K}{10}\right)^3 - \left(\frac{K-1}{10}\right)^3 & 2 \le K \le 10 \\ 0 & K > 10 \end{cases}$$

$$= \begin{cases} 0 & K<1 \\ \left(\frac{K}{10}\right)^3 - \left(\frac{K-1}{10}\right)^3 & K = 1, 2, \ldots, 10 \\ 0 & K > 10 \end{cases}$$

$\blacksquare$

next time!

compare Geometric & Exponential r.v.s

by comparing their CDFs.