CSE 107 2-6-24

3.1 Continuous Random Variables, PDFs

__Defn__

A r.v. $X : \Omega \to \mathbb{R}$ is called __continuous__ iff there exists a function

$$f_X : \mathbb{R} \to \mathbb{R}$$

called the __probability density function__ (PDF), such that for any $S \subseteq \mathbb{R}$

$$P(X \in S) = \int_S f_X(x)\,dx .$$

<u>note</u>:

- In particular if $S=[a,b]$, then

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

- integrating over $[a,b]$ and (a,b), $(a,b]$, $[a,b)$ are the same

- also if $S=\{a\}=[a,a]$ we have

$$P(X=a) = \int_a^a f_X(x)\,dx = 0$$

- to be a valid PDF, we must have

$$\int_{-\infty}^{\infty} f_X(x)\,dx = P(X \in \mathbb{R}) = P(\Omega) = 1$$

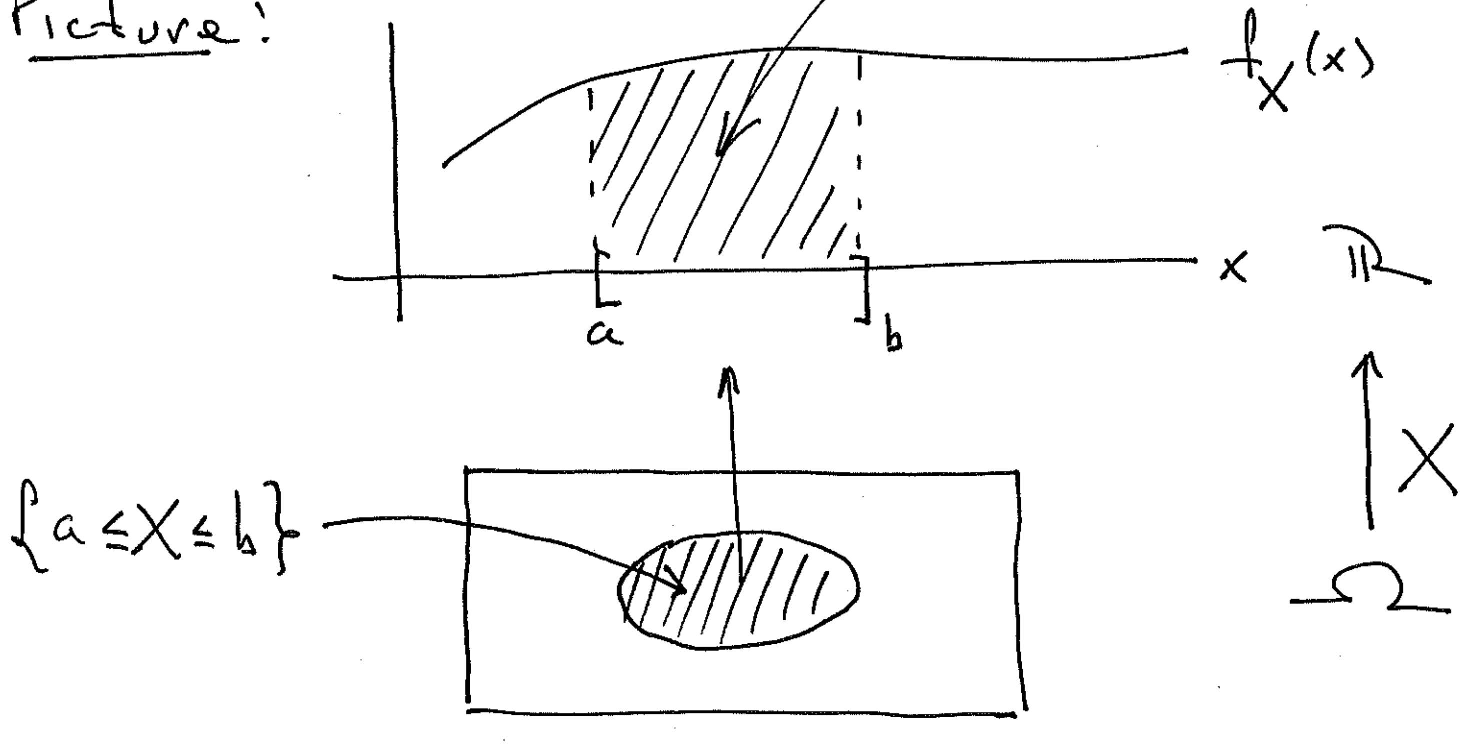
- Also: we must have

$$f_X(x) \geq 0$$

for all $x \in X$.

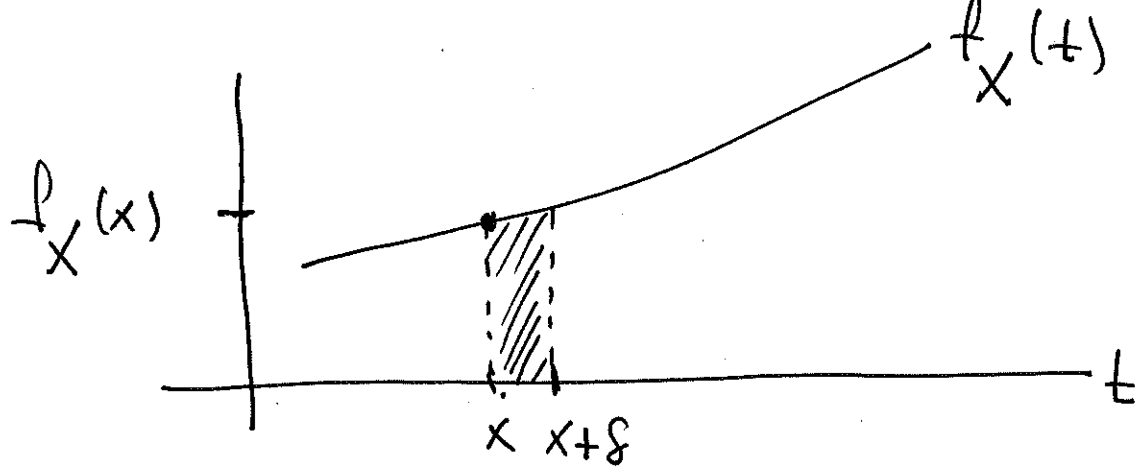
$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

Picture:



<u>note</u>: the PDF does not give the 'probability ~ mass' at any point. Instead it gives 'mass density' at a point.

let $\delta > 0$ be a small number. Then $f_X(t)$ is approx. constant on $[x, x+\delta]$.



$$\underbrace{P(x \le X \le x+\delta)}_{\text{units: 'mass'}} = \int_x^{x+\delta} f_X(t)\,dt \approx \underbrace{f_X(x)}_{} \cdot \underbrace{\delta}_{\text{length}}$$

$\therefore$ $f_X(x)$ has unit: $\frac{\text{mass}}{\text{length}}$ = density

<u>note</u>: $f_X(x)$ may have jump discontinuities

<u>Ex.</u> <u>Continuous Uniform</u> r.v. on $[a, b]$

$$f_X(x) = \begin{cases} c & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

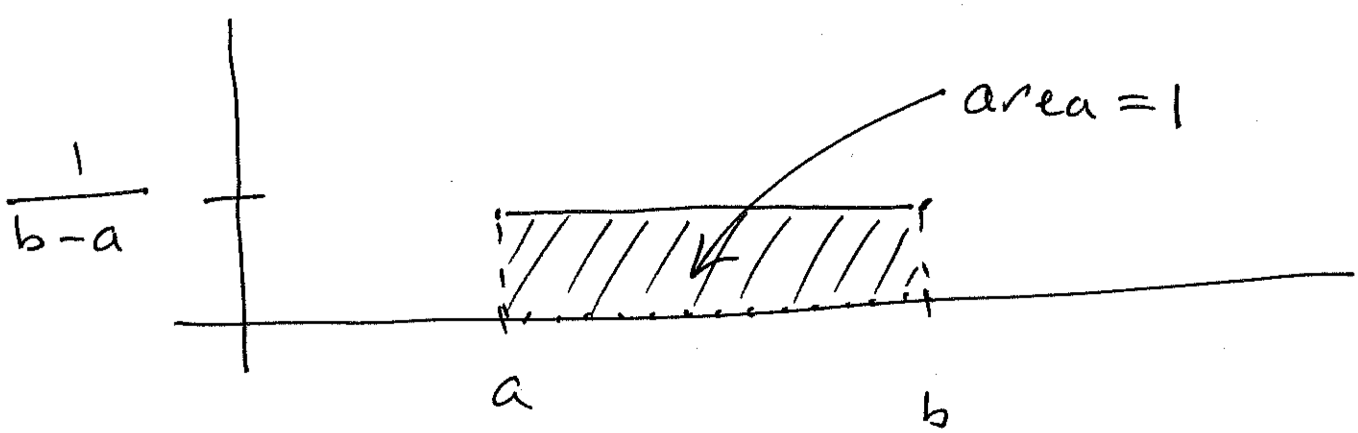
To compute c, we use:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b c\,dx = c \cdot x \Big|_a^b = c(b-a)$$

$$\therefore \quad \boxed{c = \frac{1}{b-a}}$$

$$\therefore \quad f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



Ex. Piecewise constant PDF.

Alice's driving time to work is

- 40-44 minutes in good weather.
- 44-50 " " bad " .

uniformly distributed in each case.
Suppose P(good weather) = $\frac{2}{3}$, (so P(bad weather) = $\frac{1}{3}$). Determine the PDF of Alice's driving time.

We're given

$$f_X(x) = \begin{cases} c_1 & \text{if } 40 \leq x \leq 44 \quad \text{(good)} \\ c_2 & \text{if } 44 < x \leq 50 \quad \text{(bad)} \\ 0 & \text{otherwise} \end{cases}$$

Also

$$\frac{2}{3} = P(\text{good}) = P(40 \le X \le 44)$$

$$= \int_{40}^{44} c_1 \, dx = c_1 x \Big|_{40}^{44} = c_1(44-40)$$

$$= 4c_1$$

$$\therefore \quad c_1 = \frac{1}{\cancel{4}_{2}} \cdot \frac{\cancel{2}}{3} = \boxed{\frac{1}{6}}$$

Also

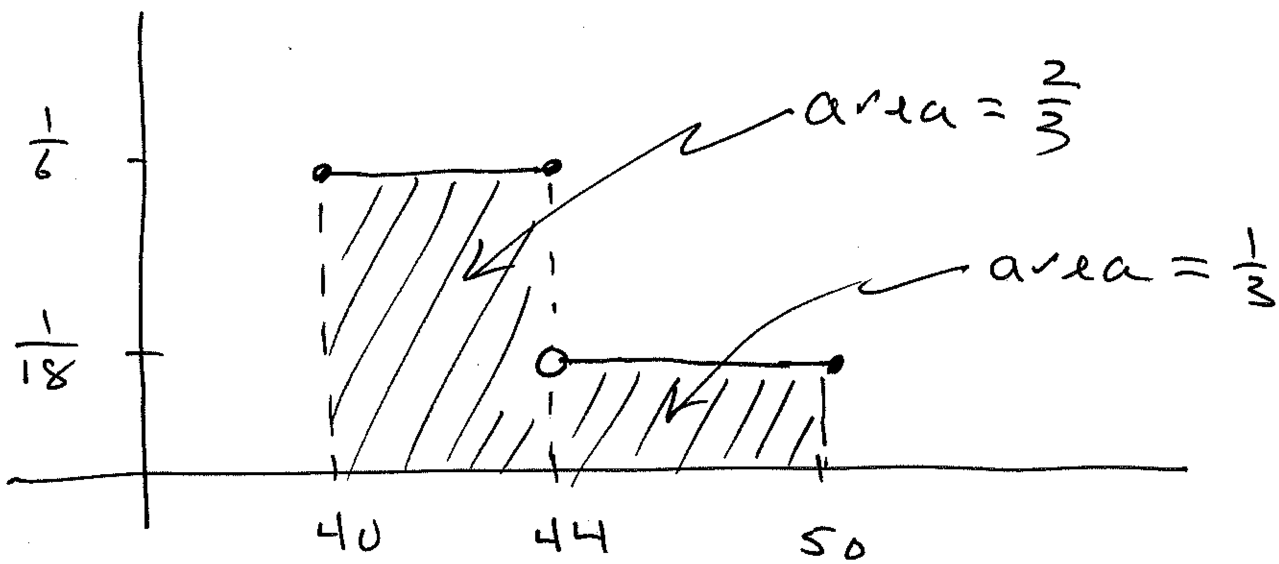
$$\frac{1}{3} = P(\text{bad}) = P(44 \le X \le 50)$$

$$= \int_{44}^{50} c_2 \, dx = c_2 \cdot (50-44) = 6c_2$$

$$\therefore \quad c_2 = \frac{1}{3} \cdot \frac{1}{6} = \boxed{\frac{1}{18}}$$

so

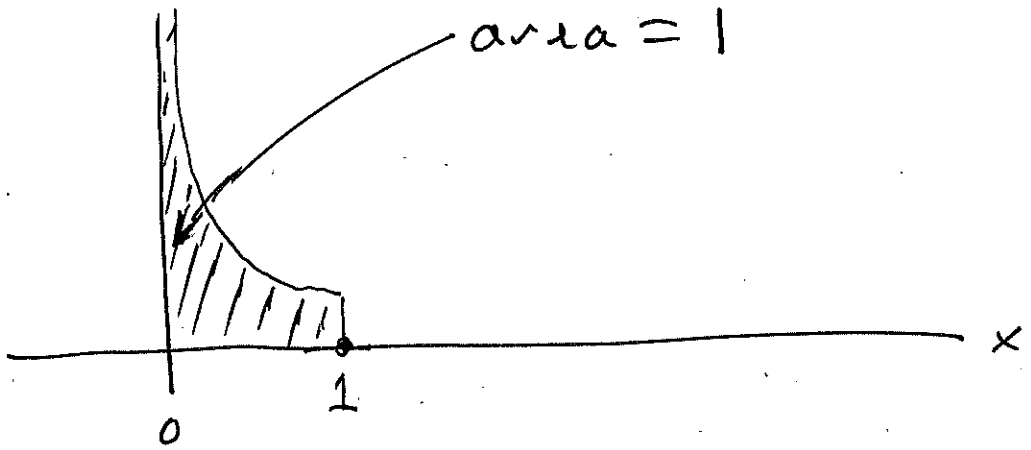
$$f_X(x) = \begin{cases} \frac{1}{6} & 40 \le x \le 44 \\ \frac{1}{18} & 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$



Ex.

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is this valid? yes!



$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int_0^1 x^{-\frac{1}{2}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{2} x^{-1/2}\,dx$$

$$= \lim_{a\to 0^+} \cancel{\frac{1}{2}} \frac{x^{1/2}}{\cancel{1/2}}\Big|_a^1 = \lim_{a\to 0^+} (1-\sqrt{a})$$

$$= 1.$$

Defn

The expected value of a cont. r.v.

X is

$$\boxed{E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$

Expected value rule

$$\boxed{E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx}$$

Proof: See Prob. 4 on p. 185 with soln.

The n^{th} moment of X

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

Variance of X:

$$\boxed{Var(X) = E[(X-E[X])^2]}$$

Theorem:

$$\boxed{Var(X) = E[X^2] - E[X]^2}$$

exercise: prove this.

If $Y = aX + b$, then

- $E[Y] = aE[X] + b$

and

- $Var(Y) = a^2 Var(X)$

exercise: prove both of these

<u>EX</u> cont. unif. r.v. on $[a,b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$$= \int_a^b x \cdot \frac{1}{(b-a)} \, dx = \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{1}{2} \cdot (b^2 - a^2)$$

$$= \frac{1}{2} \cdot \frac{1}{\cancel{b-a}} \cdot \cancel{(b-a)}(b+a)$$

$$= \frac{a+b}{2}$$

$$E[X^2] = \int_a^b x^2 \cdot \left(\frac{1}{b-a}\right) dx$$

$$= \left(\frac{1}{b-a}\right) \cdot \frac{x^3}{3} \Big|_a^b$$

$$= \frac{1}{3} \cdot \frac{1}{b-a} (b^3 - a^3)$$

$$= \frac{1}{3} \cdot \left(\cancel{\frac{1}{b-a}}\right) (\cancel{b-a})(b^2 + ab + a^2)$$

$$= \frac{a^2 + ab + b^2}{3}$$

Hence

$$Var(X) = E[X^2] - E[X]^2$$

$$= \ldots \text{exercise} \ldots$$

$$= \frac{(b-a)^2}{12}$$

Exercise: Alice's driving time

Recall

$$f_X(x) = \begin{cases} 1/6 & 40 \le x \le 44 \\ 1/18 & 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

check:

$$E[X] = \frac{131}{3}$$

$$E[X^2] = \frac{17{,}228}{9}$$

$$Var(X) = \frac{67}{9}$$

Ex. the Exponential r.v.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ a real parameter.

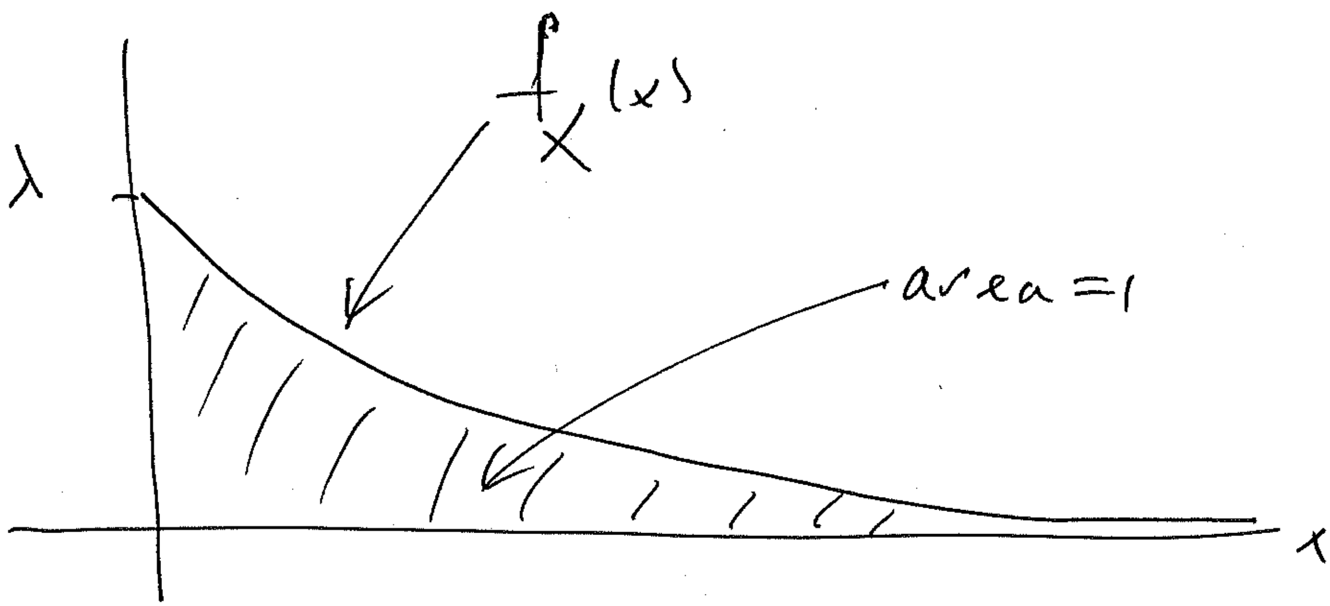
Continuous time analog of Geometric.

check

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \cancel{\lambda} \cdot \left(-\frac{1}{\cancel{\lambda}}\right) e^{-\lambda x} \Big|_0^{\infty}$$

$$= -(0 - 1) = \boxed{1}$$



$f_X(x)$
λ
area = 1
x