CSE 107 2-23-24

Supplemental Lecture

<u>Recall</u>

If X, Y are normal, then $X+Y$ is also normal. Also

$$aX + bY$$

is normal with

mean : $a\mu_X + b\mu_Y$

Var : $a^2\sigma_X^2 + b^2\sigma_Y^2$

Similarly for any number of normal random variables.

4.2 Covariance & Correlation

<u>Defn</u>

The <u>covariance</u> of r.v.s X, Y is

$$\mathrm{Cov}(X,Y) = E\left[(X-E[X])(Y-E[Y])\right]$$

observe

$$\mathrm{cov}(X,X) = \mathrm{Var}(X)$$

we say X, Y are <u>uncorrelated</u> iff

$$\mathrm{cov}(X,Y) = 0$$

Also

- if $\mathrm{cov}(X,Y) > 0$ we say <u>Positively Correlated</u>
- " $\mathrm{cov}(X,Y) < 0$ " " <u>negatively Correlated</u>

observe

$$\operatorname{cov}(X,Y)=E\big[(X-E[X])(Y-E[Y])\big]$$

$$=E\big[XY-E[X]Y-E[Y]X+E[X]E[Y]\big]$$

$$=E[XY]-E[X]E[Y]-\cancel{E[X]E[Y]}+\cancel{E[X]E[Y]}$$

Therefore

$$\boxed{\operatorname{cov}(X,Y)=E[XY]-E[X]E[Y]}.$$

Note:

$$X, Y \text{ independent} \Rightarrow E[XY] = E[X]E[Y]$$

$$\Rightarrow \mathrm{Cov}(X,Y) = 0$$

$$\Rightarrow X, Y \text{ uncorrelated.}$$

The converse is false!!

see Ex. 4.13 on P. 218–219 of text.

But contrapositive says

$$\begin{bmatrix} \text{if } X, Y \text{ pos. or} \\ \text{neg. correlated} \end{bmatrix} \Rightarrow \begin{bmatrix} X, Y \text{ are} \\ \text{not independent} \end{bmatrix}$$

<u>Defn</u>

The <u>correlation coefficient</u> $\rho(X,Y)$ is

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

$\rho(X,Y)$ is a normalized version of covariance. It can be shown

$$-1 \le \rho(X,Y) \le 1$$

max negative correlation (at -1)

max positive correlation (at 1)

Ex

we toss a coin n times where $P(\text{head}) = p$. let

$$X = \#\text{heads} \quad \text{and} \quad Y = \#\text{tails}$$

compute $\rho(X, Y)$.

solution

observe $X + Y = n$, so

$$E[X] + E[Y] = E[X+Y] = E[n] = n.$$

and

$$\begin{aligned} Var(Y) &= Var(-X + n) \\ &= (-1)^2 Var(X) \\ &= Var(X). \end{aligned}$$

also

$$(X - E[X]) + (Y - E[Y]) = n - n = 0$$

$$\therefore \quad Y - E[Y] = -(X - E[X])$$

Thus

$$\mathrm{Cov}(X,Y) = E[-(X-E[X])^2]$$

$$= -E[(X-E[X])^2]$$

$$= -\mathrm{Var}(X),$$

and

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\cdot \mathrm{Var}(Y)}}$$

$$= \frac{-\mathrm{Var}(X)}{\sqrt{\mathrm{Var}(X)^2}}$$

$$= \frac{-\operatorname{Var}(X)}{\operatorname{Var}(X)}$$

$$= \boxed{-1}$$

$\therefore$ X, Y are perfectly neg. correlated.

Variance of a Sum

Let X, Y be r.v.s (not necessarily independent.) Then

$$\operatorname{Var}(X+Y)$$

$$= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y)$$

In general for: $X_1, X_2, \ldots, X_n$

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{\substack{i,j \\ i \neq j}} \mathrm{Cov}(X_i, X_j)$$

Proof of case $n=2$:

Let $U = X - E[X] \quad \therefore E[U] = 0$

$V = Y - E[Y] \quad \therefore E[V] = 0$

Thus

$$\mathrm{Var}(X+Y) = \mathrm{Var}(U+V)$$

$$= E\left[\left((U+V) - E[U+V]\right)^2\right]$$

$$= E\left[(U+V)^2\right]$$

$$= E\left[U^2 + V^2 + 2UV\right]$$

$$= E[U^2] + E[V^2] + 2E[UV]$$

$$= Var(X) + Var(Y) + 2\,Cov(X, Y)$$

$\blacksquare$

See Ex. 4.15 on P. 221.

4.3 Conditional Expectation & Variance

Recall

$$g(y) = E[X \mid Y = y]$$

is a function of y. If we substitute Y for y in g we get a new r.v.

$$g(Y) = E[X \mid Y].$$

The expected value of $g(Y)$ is denoted

$$E[g(Y)] = E[E[X \mid Y]]$$

By the expected value rule

$$E[E[X|Y]] = E[g(Y)]$$

$$= \int_{-\infty}^{\infty} g(y) \cdot f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\,dy$$

$$= E[X]$$

where last eqn. is the total expectation thm. We call this the law of iterated expectation

$$\boxed{E[E[X|Y]] = E[X]}$$

We can do something similar with variance. First define a function

$$h(y) = Var(X \mid Y=y)$$

$$= E[(X - E[X \mid Y=y])^2 \mid Y=y]$$

By substituting r.v. Y for y we get a new r.v.

$$h(Y) = Var(X \mid Y)$$

$$= E[(X - E[X \mid Y])^2 \mid Y]$$

We call this the <u>conditional variance</u> of X given Y.

we have

<u>law of total Variance</u>

$$\boxed{Var(X) = E[Var(X|Y)] + Var(E[X|Y])}$$

<u>Proof</u>

By variance in terms of moments formula, we have

$$Var(X|Y) = E[X^2|Y] - E[X|Y]^2$$

$$\therefore E[X^2|Y] = Var(X|Y) + E[X|Y]^2$$

By law of iterated expectation we have

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= E[E[X^2|Y]] - E[E[X|Y]]^2$$

$$= E[\mathrm{Var}(X|Y) + E[X|Y]^2]$$

$$- E[E[X|Y]]^2$$

$$= E[\mathrm{Var}(X|Y)]$$

$$+ E[E[X|Y]^2] - E[E[X|Y]]^2$$

for any $g: \mathbb{R} \to \mathbb{R}$ we have

$$\mathrm{Var}(g(Y)) = E[g(Y)^2] - E[g(Y)]^2$$

letting $g(Y) = E[X|Y]$, we have

$$Var(E[X|Y]) = E[E[X|Y]^2] - E[E[X|Y]]^2$$

Therefore

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]).$$

$$\blacksquare$$