CSE 107 2-22-24

lab stuff:

To get file output in Linux:

$ prog > file

then convert file to a pdf.

Thm

if X is a cont. r.v., $Y = g(X)$,

g is strictly monotonic (inc. or dec.),

and $h = g^{-1}$, then

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| .$$

Proof:

case ①: g is increasing

Then $h = g^{-1}$ is also increasing,

so that $h'(y) > 0$ for all y.

$$F_Y(y) = P(Y \le y)$$

$$= P(g(X) \le y)$$

$$= P(X \le h(y))$$

$$= F_X(h(y))$$

Therefore, by diff. w.r.t y:

$$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

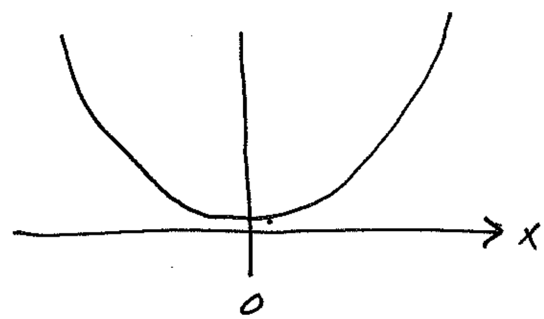
$$= f_X(h(y)) \cdot |h'(y)|.$$

exercise: do <u>case 2</u>: <u>g</u> is <u>decreasing</u>

$\blacksquare$

Ex. Suppose $Y = X^2$ where X is a cont. r.v.. Find f_Y in terms of f_X.

solution $g(x) = x^2$



$g : [0,\infty) \to [0,\infty)$ is strictly monotone (inc.) with $g^{-1}(y) = \sqrt{y}$.

note: we assume $\text{range}(X) \subseteq [0,\infty)$ so $\text{range}(Y) \subseteq [0,\infty)$

Thus

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \quad (y \geq 0)$$

<u>Two random variables</u>: $Z = g(X, Y)$

<u>Ex</u> let X, Y be cont. uniform <u>independent</u> r.v.s on $[0,1]$. Thus

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

similarly for Y.

Define Z by

$$Z = \max(X, Y)$$

Find $f_Z(z)$.

For $0 \le z \le 1$ we have

$$F_Z(z) = P(Z \le z)$$

$$= P(\max(X, Y) \le z)$$

$$= P(X \le z, Y \le z)$$

$$= P(X \le z) \cdot P(Y \le z) \quad \text{by indep.}$$

$$= F_X(z) \cdot F_Y(z)$$

$$= z^2$$

Hence

$$f_Z(z) = \begin{cases} 2z & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Exercise</u>

Let X, Y be independent, cont. uniform r.v., where X is unif. on $[a, b]$ and Y is unif on $[c, d]$

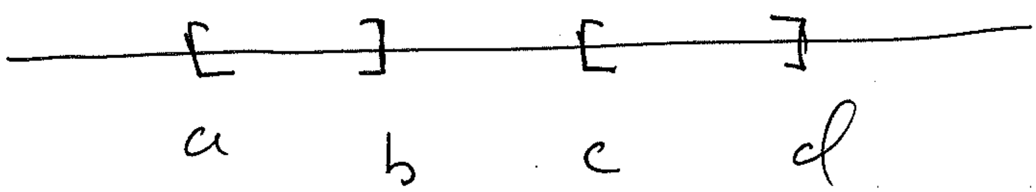
Define: $Z = \max(X, Y)$

Find the PDF of Z.

<u>note</u>:

- if $b < c$ we have $[a,b] \cap [c,d] = \phi$.

$$a < b < c < d$$



so $Z = \max(X, Y) = Y$

- similarly if $d < a$ we have

$$Z = \max(X, Y) = X$$

- Assume $c \le b$ and $a \le d$ for a non-trivial problem.

Sums and convolutions

Let X, Y be indep. discrete r.v.s with P_X, P_Y as PMFs.

Let

$$Z = X + Y$$

Goal: find $P_Z(z)$.

we have for $z \in \text{range}(Z)$

$$p_Z(z) = P(X+Y=z)$$

$$= \sum_{\{(x,y) \mid x+y=z\}} P(X=x, Y=y)$$

$$= \sum_{\{(x,z-x) \mid x \in \text{range}(X)} P(X=x) \cdot P(Y=z-x) \quad \{\text{by indep.}$$

$$= \boxed{\sum_x p_X(x) \cdot p_Y(z-x)}$$

This operation is called the <u>convolution product</u> of p_X and p_Y.

<u>notation</u> $p_Z = p_X * p_Y$

<u>Defn</u>

$$(f * g)(z) = \sum_{x \in dom(f)} f(x) \cdot g(z-x)$$

Now let X, Y be indep. (jointly) cont. r.v. and

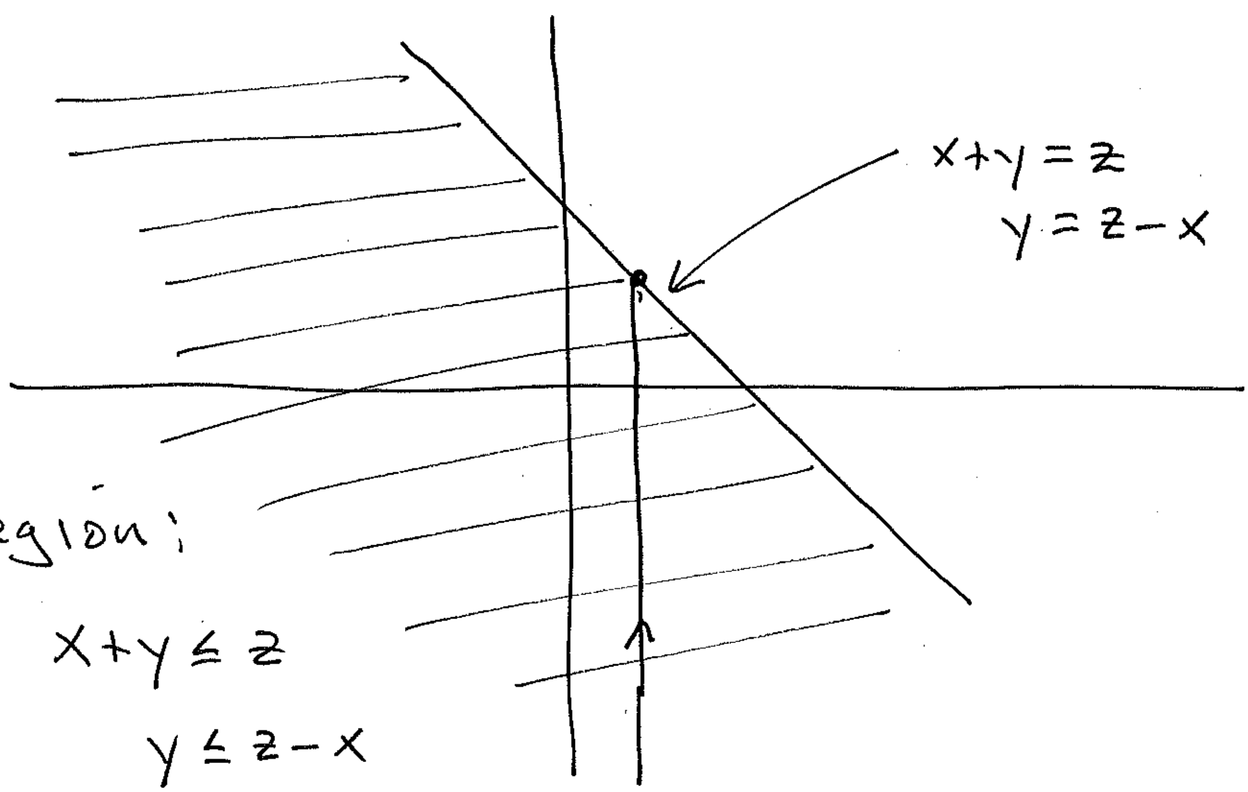
$$Z = X + Y$$

Then

$$F_Z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$

$$= \iint_{\{(x,y) \mid x+y \le z\}} f_{X,Y}(x,y)\,dy\,dx$$



Region:

$x + y \le z$

$y \le z - x$

so

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) \cdot f_Y(y) \, dy \, dx \quad \left\{ \begin{matrix} \text{by} \\ \text{indep.} \end{matrix} \right.$$

$$= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{z-x} f_Y(y) \, dy \right) dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \, F_Y(z-x) \, dx$$

Now differentiate w.r.t z:

$$f_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x) \, F_Y(z-x) \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \frac{d}{dz} F_Y(z-x)\, dx$$

$$= \boxed{\int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx}$$

This operation is called the <u>convolution product</u> of f_X and f_Y

<u>notation</u>: $(f_X * f_Y)(z)$

i.e.

$$(f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx$$

i.e. $f_Z = f_X * f_Y$

i.e. $$f_{X+Y} = f_X * f_Y$$

$$= f_Y * f_X$$

<u>note</u>: must have X, Y indep. if we use convolution.

Ex.

let X, Y be indep., cont. uniform on $[a, b]$, i.e.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and similarly for f_Y. let

$$Z = X + Y$$

Then

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

The integrand is $\boxed{\frac{1}{(b-a)^2}}$ iff

both

$$a \le x \le b \quad \text{and} \quad a \le z - x \le b$$

i.e.

$$a \le x \le b \quad \text{and} \quad z-b \le x \le z-a$$

i.e. integrand is non-zero iff

$$\max(a, z-b) \le x \le \min(b, z-a)$$

Thus

$$f_Z(z) = \int_{\max(a,z-b)}^{\min(b,z-a)} \frac{1}{(b-a)^2}\, dx$$

$$\therefore f_Z(z) = \frac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2}$$

The interval of integration is empty if either

$$z - a < a \Rightarrow z < 2a$$

or

$$z - b > b \Rightarrow z > 2b$$

$$f_Z(z) = \begin{cases} \dfrac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2} & 2a \le z \le 2b \\ 0 & \text{otherwise} \end{cases}$$

observe: $Z = X + Y$ is <u>not</u> uniform.

Ex.

let X, Y be indep. normal r.v.'s with

mean: μ_X, μ_Y resp.

and

vars: σ_X^2, σ_Y^2 resp.

Thus

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot e^{-(x-\mu_X)^2/2\sigma_X^2}$$

and similarly for Y.

let $Z = X + Y$.

Then

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\cdot\sigma_X} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y} \cdot e^{\frac{-(z-x-\mu_Y)^2}{2\sigma_Y^2}} \, dx$$

$$= \text{---- (exercise) -----}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{\sigma_X^2+\sigma_Y^2}} \cdot e^{\frac{-(z-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}}$$

$\therefore$ Z is normal with

mean: $\mu_X + \mu_Y$

Var : $\sigma_X^2 + \sigma_Y^2$