3.6 Bayes Rule

Recall:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

also: if $A_1, \ldots, A_n$ form a partition of Ω, then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

Bayes Rule:

$$P(A_i|B) = \frac{P(B|A_i)\cdot P(A_i)}{\sum_{j=1}^{n} P(B|A_j)\cdot P(A_j)}$$

If X, Y are discrete r.v., this becomes

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)\cdot P_X(x)}{\sum_{t} P_{Y|X}(y|t)\cdot P_X(t)}$$

sum over $t \in \text{range}(X)$

In the continuous case, this is

$$f_{X|Y}(x|y) = \frac{f_X(x) \cdot f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t) f_{Y|X}(y|t)\, dt}$$

Ex:

Let Y be a normal r.v. with s.d. 1 and mean X, another r.v. Suppose X is cont. uniform on $[1, 3]$.

a.) find $f_Y(y)$

we are given

$$f_X(x) = \begin{cases} \frac{1}{2} & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2} & \text{if } 1 \le x \le 3,\ y \in \mathbb{R} \\ 0 & \text{otherwise},\ y \in \mathbb{R} \end{cases}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_1^3 f_{Y|X}(y|x) \cdot f_X(x)\,dx$$

$$= \int_1^3 \frac{1}{2} \cdot e^{-(y-x)^2/2} \cdot \frac{1}{\sqrt{2\pi}}\,dx$$

$$\left\{ \begin{array}{l} \text{let} \\ u = x - y \\ du = dx \end{array} \right.$$

$$= \frac{1}{2} \int_{1-y}^{3-y} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

$$= \frac{1}{2}\left(\Phi(3-y) - \Phi(1-y)\right)$$

b.) find $f_{X|Y}(x|y)$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)} & 1 \le x \le 3 \\ & y \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

c.) suppose we sample Y and get $Y = y = 3$. What is the probability that $X \le 2$?

answer = $\boxed{.2848}$

d.) find $E[Y]$

answer = $\boxed{2}$

4.1 Derived distributions

let $Y = g(X)$. how can we compute

- cont. case: $f_Y(y)$

from $f_X(x)$? we call $f_Y(y)$ a <u>derived distribution</u>.

<u>2-step process</u>:

① find CDF $F_Y(y)$

$$F_Y(y) = P(g(X) \le y) = \int_{\{x \mid g(x) \le y\}} f_X(x)\,dx$$

(2) differentiate

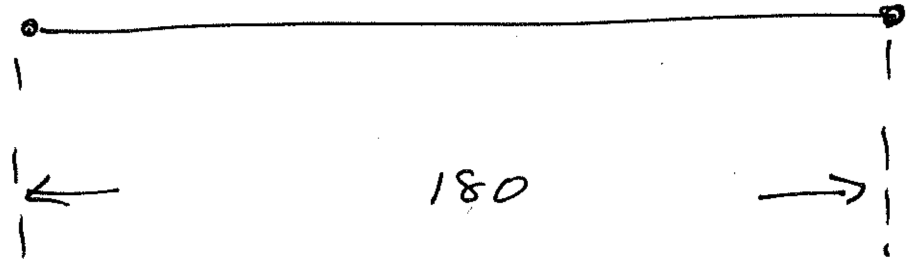
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Ex.

A car travels 180 miles at const. rate R, a random variable.

Suppose R is uniform on

$30 \le r \le 60$ (mph)



Let T be time of travel.

find $f_T(t)$.

solution

from $180 = R \cdot T$ we have

$$T = \frac{180}{R}$$

① $F_T(t) = P(T \le t)$

$$= P\left(\frac{180}{R} \le t\right)$$

$$= P\left(R \ge \frac{180}{t}\right)$$

$$= 1 - P\left(R < \frac{180}{t}\right)$$

$$= 1 - F_R\left(\frac{180}{t}\right)$$

② diff both sides w.r.t. t !

$$f_T(t) = -f_R\left(\frac{180}{t}\right)\cdot\left(-\frac{180}{t^2}\right)$$

we are given

$$f_R(r) = \begin{cases} \frac{1}{30} & 30 \le r \le 60 \\ 0 & \text{otherwise} \end{cases}$$

hence

$$f_T(t) = \begin{cases} -\frac{1}{30}\cdot\left(-\frac{180}{t^2}\right) & 30 \le \frac{180}{t} \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{6}{t^2} & \dfrac{1}{60} \le \dfrac{t}{180} \le \dfrac{1}{30} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{6}{t^2} & \text{if } 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

∎

If $Y = aX + b$, then

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)} \quad (a \neq 0)$$

<u>Proof</u>

$$F_Y(y) = P(Y \le y)$$

$$= P(aX + b \le y)$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a > 0 \\ P\left(X \ge \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a > 0 \\ 1 - P\left(X < \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

so

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right).$$

∎

<u>Ex.</u>

Let X be normal with mean μ and variance σ^2. Then

$$Y = aX + b$$

is also normal with

mean: $a\mu + b$

variance: $a^2\sigma^2$

Proof

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

Thus

$$f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{\sqrt{2\pi} \cdot |a| \cdot \sigma} \cdot e^{-\left(\frac{y-b}{a} - \mu\right)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,|a|\sigma} \cdot e^{-\left(\frac{y-b-a\mu}{a}\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,|a|\sigma} \cdot e^{-(y-(a\mu+b))^2 / 2(a\sigma)^2}$$

which is normal with

mean : $a\mu + b$

Variance : $a^2\sigma^2$

$\blacksquare$

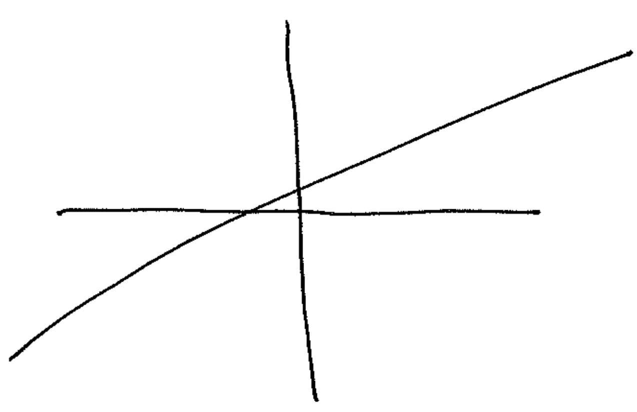
Defn

a function $g : \mathbb{R} \to \mathbb{R}$ is called strictly monotonic iff for all $x_1, x_2 \in \mathbb{R}$

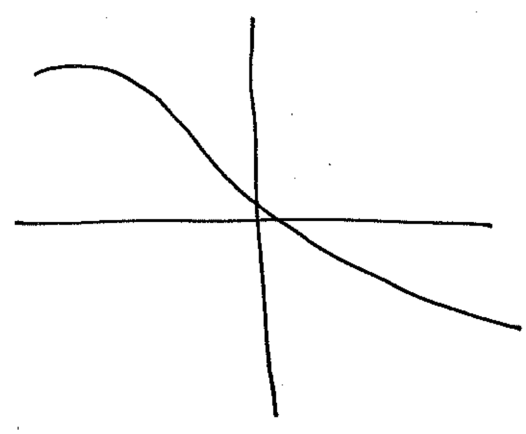
(1) $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$ (increasing)

or

(2) $x_1 < x_2 \Rightarrow g(x_1) > g(x_2)$ (decreasing)



(1)



(2)

also assume g is surjective. Such a function is necessarily bijective, and therefore invertible.

Let $h = g^{-1}$, i.e.

$$h(g(x)) = x \quad \text{for all } x \in \mathbb{R},$$

and

$$g(h(y)) = y \quad \text{for all } y \in \mathbb{R}.$$

It can be shown (exercise)

(1) $\Rightarrow$ $h = g^{-1}$ is monotone increasing

and

(2) $\Rightarrow$ $h = g^{-1}$ " " decreasing.

Now let X be a cont. r.v.

and

$$Y = g(X)$$

If g is differentiable, then so is $h = g^{-1}$ and

$$(1) \Leftrightarrow g' > 0 \Leftrightarrow h' > 0 .$$

and

$$(2) \Longleftrightarrow g' < 0 \Longleftrightarrow h' < 0 .$$

Theorem

$$\boxed{f_Y(y) = f_X(h(y)) \cdot |h'(y)|}$$

Proof: exercise.