CSE 107 2-16-24

Supplemental Lecture

Conditional Expectation

Definitions

- $E[X|A] = \int_{-\infty}^{\infty} x \, f_{X|A}(x) \, dx$

- $E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$

Expected Value rule

- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) \, f_{X|A}(x) \, dx$

- $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) \, f_{X|Y}(x|y) \, dx$

total expectation theorem

- If $A_1, A_2, \dots, A_n$ form a partition of Ω, and $P(A_i) > 0$ for all i, then

$$(1) \quad E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]$$

- If $P(Y=y) > 0$ for all $y \in \mathbb{R}$, then

$$(2) \quad E[X] = \int_{-\infty}^{\infty} f_Y(y) \cdot E[X \mid Y=y] \, dy$$

Proof of (1)

start with total probability theorem

$$f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)$$

multiply by x, integrate w.r.t. x, to get $E[X]$

$$\int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \sum_{i=1}^{n} P(A_i)\cdot f_{X|A_i}(x)\,dx$$

$$\therefore\ E[X] = \sum_{i=1}^{n} P(A_i)\cdot \int_{-\infty}^{\infty} x f_{X|A_i}(x)\,dx$$

$$= \sum_{i=1}^{n} P(A_i) E[X|A_i] \quad \blacksquare$$

<u>Proof of (2)</u>

start with the RHS:

$$\int_{-\infty}^{\infty} E[X|Y=y]\cdot f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx \right)\cdot f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \cdot f_Y(y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, f_Y(y) \, dy \, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{X,Y}(x,y) \, dy \, dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy \right) dx$$

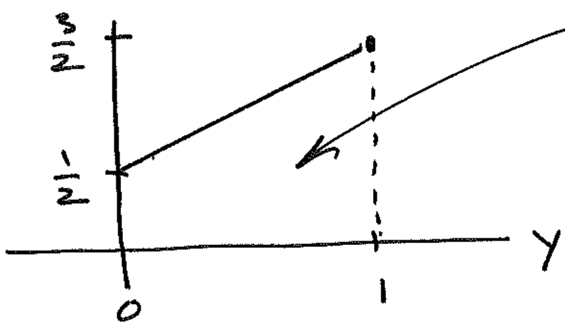
$$= \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx = E[X],$$

which is the RHS. $\blacksquare$

Ex.

let X, Y be jointly cont. with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



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and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X]$.

Solution

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$= \int_0^1 x \left(\frac{x+y}{y+\frac{1}{2}}\right) dx$$

$$= \frac{1}{y+\frac{1}{2}} \cdot \int_0^1 x^2 + xy \, dx = \frac{1}{y+\frac{1}{2}} \cdot \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\Big|_0^1$$

$$= \frac{\frac{1}{2}y + \frac{1}{3}}{y+\frac{1}{2}} \quad \text{for} \quad 0 \le y \le 1$$

Thus

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\, dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}y + \frac{1}{3}}{\cancel{y+\frac{1}{2}}}\right) \cdot \cancel{(y+\tfrac{1}{2})}\, dy$$

$$= \frac{1}{2} \cdot \frac{1}{2} y^2 + \frac{1}{3} y \Big|_0^1 = \frac{1}{4} + \frac{1}{3} = \boxed{\frac{7}{12}}$$

Another way:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot f_Y(x)$$

$$= \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy = xy + \tfrac{1}{2}y^2 \Big|_0^1$$

$$= \begin{cases} x+\frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore\ E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_0^1 x\left(x+\tfrac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^2 + \tfrac{1}{2}x\right) dx$$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{4}x^2 \Big|_0^1 = \tfrac{1}{3} + \tfrac{1}{4} = \boxed{\tfrac{7}{12}}.$$

Backing up in this example: recall $E[X \mid Y=y]$ is a function of y!

$$E[X \mid Y=y] = \begin{cases} \dfrac{\frac{1}{2}y + \frac{1}{3}}{y + \frac{1}{2}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Call this $g(y)$:

$$g(y) = E[X \mid Y=y]$$

We can compose this function with Y:

$$g(Y) = E[X \mid Y]$$

to get a r.v.

check that the mean of this r.v. is

$$E[g(Y)] = E[E[X|Y]]$$

$$= \int_0^1 y\left(\frac{\frac{1}{2}y + \frac{1}{3}}{y + \frac{1}{2}}\right) dy$$

$$= \text{---- exercise ----}$$

$$= \boxed{\frac{7}{12}}$$

In Chap. 4 we'll write

$$E[E[X|Y]] = E[X]$$

called: <u>law of iterated expectations</u>

Independence

let X, Y be jointly continuous r.v.s

Defn

we say X, Y are independent iff

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

for all $x \in$ range(X), $y \in$ range(Y).

Equivalently

$$\boxed{f_{X|Y}(x|y) = f_X(x) \quad (\text{if } f_Y(y) > 0)}$$

or

$$\boxed{f_{Y|X}(y|x) = f_Y(y) \quad (\text{if } f_X(x) > 0)}$$

If X, Y are indep., then any events definable in terms of X and Y are independent:

$$\{X \in A\} \text{ and } \{Y \in B\}$$

for $A, B \subseteq \mathbb{R}$. In particular

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\, ds\, dt$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_X(s) \cdot f_Y(t)\, ds\, dt$$

$$= \int_{-\infty}^{x} f_X(s)\, ds \cdot \int_{-\infty}^{y} f_Y(t)\, dt$$

$$= P(X \le x) \cdot P(Y \le y)$$

$$= F_X(x) \cdot F_Y(y).$$

In fact, converse is also true.

Thus

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$$

is equivalent to independence of X, Y.

As in the discrete case, we have, for indep. X, Y continuous r.v.s

(1) $E[XY] = E[X] \cdot E[Y]$

(2) X, Y indep. $\Longrightarrow g(X), h(Y)$ indep.
so $E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$.

(3) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Exercise:

- Prove (1) in cont. case
- Prove (2) by adapting Problem #44 (chap. 2, p. 134) to continuous case
- Prove (3) in cont. case.