CSE 107 2-15-24

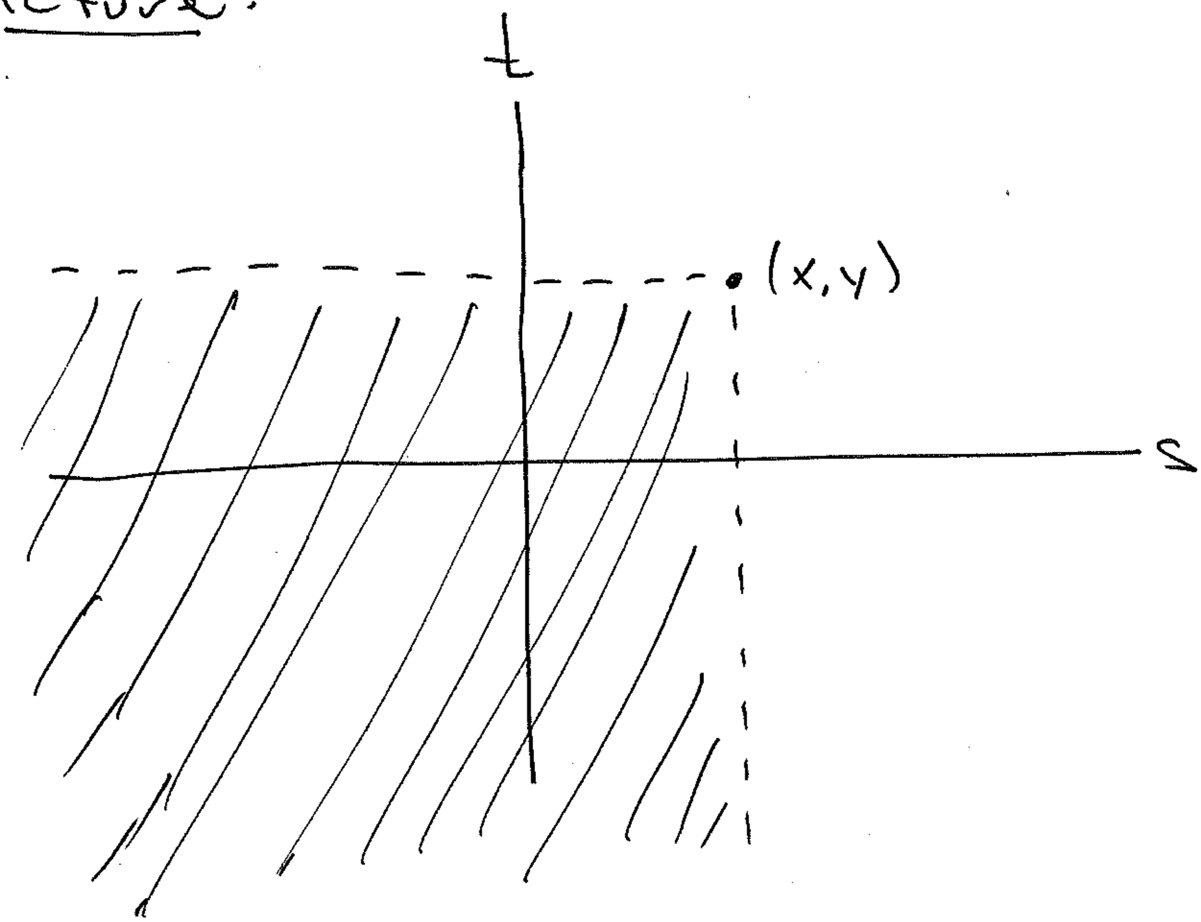
Joint CDFs

If X, Y are two r.v.s, their Joint CDF is

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

if X,Y Jointly continuous $\longrightarrow$ $$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds$$

Picture:



We can recover Joint PDF by

$$f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial y \partial x}(x,y)$$

Expectation

If $g : \mathbb{R}^2 \to \mathbb{R}$, we have the 2-variable expected value rule

$$E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

special cases:

$$E[aX + bY + c]$$

$$= \iint_{\mathbb{R}^2} (ax+by+c) f_{X,Y}(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

$$+ b\int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$+ c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty} x\, f_X(x)\,dx + b\int_{-\infty}^{\infty} y\, f_Y(y)\,dy + c$$

$$= aE[X] + bE[Y] + c$$

We can do all this for 3, 4, 5, ...
of r.v.s ... (exercise)

3.5 Conditioning

As before we begin by conditioning on a single event A, where $P(A) > 0$. Let X be a cont. r.v.

Defn

The conditional PDF of X given A is the function $f_{X|A}(x)$ satisfying

$$P(X \in S \mid A) = \int_S f_{X|A}(x)\,dx$$

for 'any' subset $S \subseteq \mathbb{R}$.

It is required that

$$\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = 1$$

EX.

The time T until a new light bulb burns out is exponential with parameter λ. Alice turns on the light, leaves and returns t hours later, find the light still on. Thus, the event

$$A = \{T > t\}$$

has occurred. Let X be the additional time until burn-out.

find the cond. PDF

$$f_{X|A}(x)$$

<u>solution</u>

we 1st find the cond. CDF

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= \int_{-\infty}^{x} f_{X|A}(s)\,ds\ ,$$

then differentiate to get

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x)$$

Recall

$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

Thus for $x \geq 0$ we have

$$P(X > x \mid A) = P(T > t + x \mid T > t)$$

$$= \frac{P(T > t+x,\ T > t)}{P(T > t)}$$

$$= \frac{P(T > t+x)}{P(T > t)}$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}}$$

$$= \frac{\cancel{e^{-\lambda t}} \cdot e^{-\lambda x}}{\cancel{e^{-\lambda t}}} = e^{-\lambda x}$$

Therefore

$$F_{X|A}(x) = P(X \leq x \mid A)$$

$$= 1 - P(X > x \mid A)$$

$$= 1 - e^{-\lambda x} \quad (x \geq 0)$$

i.e

$$F_{X|A}(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Hence

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Therefore the additional time t burn out, after t hours of use, is exponential with same parameter λ.

Thus the exponential r.v. has no memory. i.e.

$$f_{X|A}(x) = f_T(x) \quad (x \geq 0)$$

<u>Exercise</u>

Do same thing for the Geometric.

$$\overbrace{\underbrace{0\;0\;0\;\cdots\;0}_{m}\;\underbrace{0\;0\;\cdots\;0\;0\;1}_{X}}^{Y}$$

let

$Y = \#$ flips to 1st head

$X = \#$ flips after m^{th} to 1st head

and

$$A = \{ Y > m \}.$$

show:
$$p_{X|A}(k) = p_Y(k) \qquad (k = 1, 2, \ldots)$$

If events $A_1, A_2, \ldots, A_n$ form a partition of Ω, the total prob. thm gives

$$P(X \le x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \le x \mid A_i)$$

Thus

$$F_X(x) = \sum_{i=1}^{n} P(A_i) \, F_{X|A_i}(x)$$

Differentiating w.r.t x gives

$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)}$$

<u>Defn</u>

if X, Y are cont. r.v.s and $f_Y(y) > 0$ for all $y \in \mathbb{R}$, the <u>conditional PDF of X given Y</u> is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

observe

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \frac{1}{f_Y(y)} \cdot \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \frac{1}{\cancel{f_Y(y)}} \cdot \cancel{f_Y(y)} = 1 \quad \checkmark$$

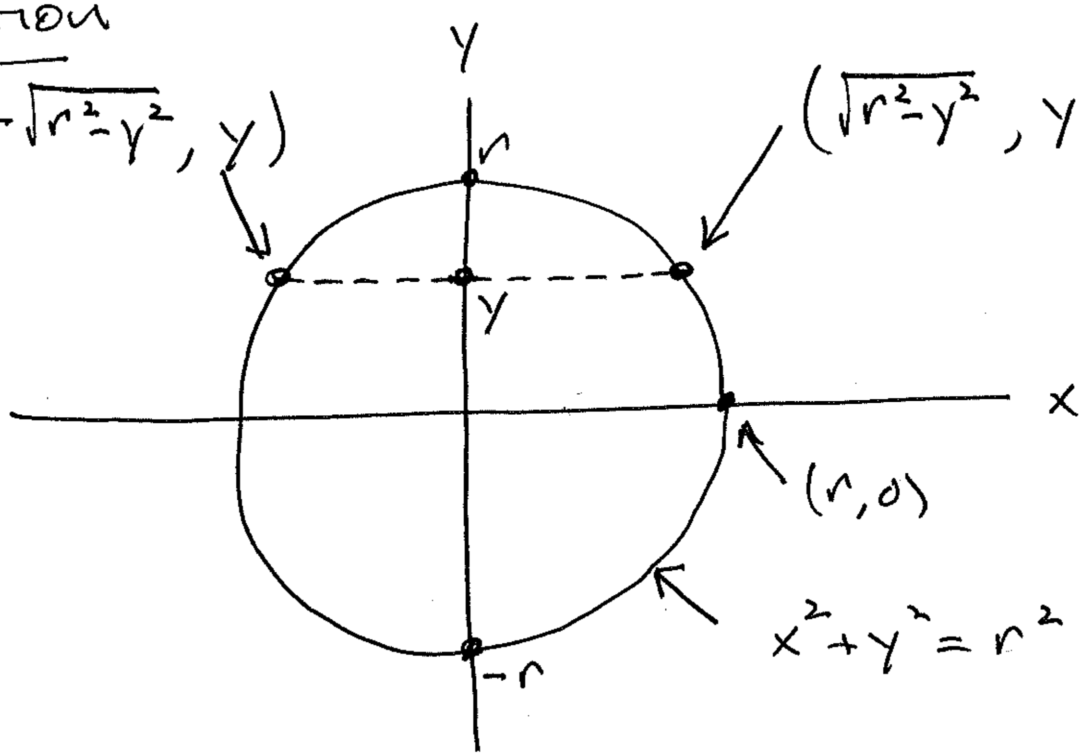
<u>EX.</u>

Throw a dart at a circular dart board of radius r, centered at $(0,0)$. Assume target is certain to be hit and each (x,y) is equally likely, i.e. uniformly distributed.

Let (X, Y) be coordinates of impact point. Find

$f_{X,Y}(x,y)$, $f_Y(y)$, $f_{X|Y}(x|y)$

<u>Solution</u>



we are given

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2} \times \Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}$$

$$= \frac{1}{\pi r^2} \cdot 2\sqrt{r^2-y^2}$$

Thus

$$f_Y(y) = \begin{cases} \dfrac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } |y| \le r \\ 0 & \text{otherwise} \end{cases}$$

Finally

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{r^2-y^2}} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

$\blacksquare$