Thm

If $f_X(x) = f_X(-x) \quad \forall x \in \mathbb{R}$,

then $E[X] = 0$.

Proof.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{0} x f_X(x)\,dx}_{} + \int_{0}^{\infty} x f_X(x)\,dx$$

$t = -x \quad dt = -dx$

$x = -t \quad dx = -dt$

$x \le 0 \Leftrightarrow t \ge 0$

$x = 0 \Leftrightarrow t = 0$

$x \to -\infty \Leftrightarrow t \to \infty$

$$= \int_{\infty}^{0} (\cancel{-}t) f_X(-t)(\cancel{-}1)\,dt \;+\; ''$$

$$= -\int_{0}^{\infty} t f_X(t)\,dt + \int_{0}^{\infty} x f_X(x)\,dx$$

$$= 0$$

$\blacksquare$

<u>corollary</u> if $f_X(x-\mu) = f_X(-(x-\mu))$,

then $E[X] = \mu$. ✓

<u>Proof</u>. Let $Y = X - \mu$. then

$E[Y] = 0$ by last thm, so

$E[X-\mu] = 0 \;\therefore\; E[X] - \mu = 0$

$\blacksquare$

Ex.

The mean high temp in S.C. on Nov. 1 is $66^\circ F$ with a std. dev. of 4.4°. What is the prob. that the high temp. X is $\leq 57^\circ$.

soln

Let X be the high temp. Then

$$P(X \leq 57) = P\left(\frac{X-66}{4.4} \leq \frac{57-66}{4.4}\right)$$

$$= P(Y \leq -2.04)$$

$$= \Phi(-2.04)$$

$$= 1 - \Phi(2.04)$$

$= 1 - .9793$

$= \boxed{.0207}$

Ex.

What is the prob. that a normal r.v. X is within 1 std. dev. of its mean?

Solution

$$P(\mu - \sigma \le X \le \mu + \sigma)$$

$$= P\left(\frac{(\mu-\sigma)-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{(\mu+\sigma)-\mu}{\sigma}\right)$$

$$= P(-1 \le Y \le 1)$$

$$= P(Y \le 1) - P(Y < -1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - (1 - \Phi(1))$$

$$= 2\,\Phi(1) - 1$$

$$= 2(.8413) - 1 = \boxed{.6826}$$

similarly:

- 2 std. devs. from mean

$$2\,\Phi(2) - 1 = 2(.9772) - 1 = \boxed{.9544}$$

- 3 std. devs from mean

$$2\,\Phi(3) - 1 = 2(.9987) - 1 = \boxed{.9974}$$

3.4 Joint PDFs of several r.v.s

Defn

We say X, Y are <u>jointly continuous</u> iff there exists $f_{X,Y}(x,y)$ such that

$$P\big((X,Y)\in B\big) = \iint_B f_{X,Y}(x,y)\,dx\,dy$$

for any subset $B \subseteq \mathbb{R}^2$. We call $f_{X,Y}(x,y)$ the <u>joint PDF</u> of X, Y.

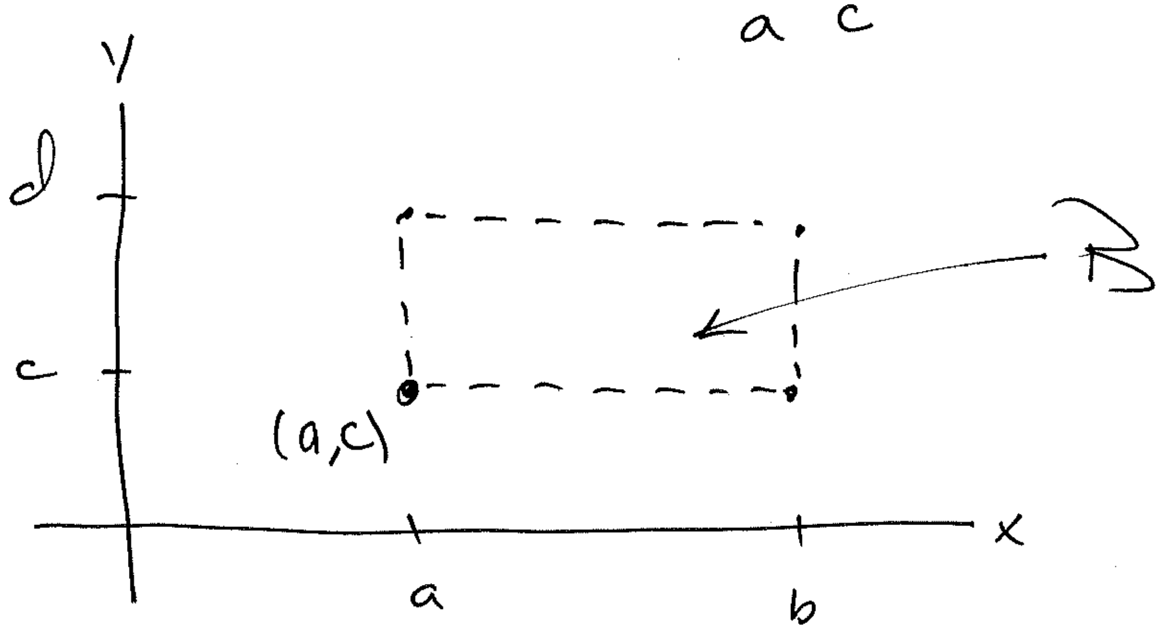
In Particular, if

$$B = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$$

then

$$P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$$



If $B = \mathbb{R}^2$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = P(\Omega) = 1$$

If $\delta > 0$ is small, and $f_{X,Y}(x,y)$ is cont. near (a,c), then

$$P(a \le X \le a+\delta,\ c \le Y \le c+\delta)$$

$$= \int_a^{a+\delta}\int_c^{c+\delta} f_{X,Y}(x,y)\,dy\,dx$$

$$\approx f_{X,Y}(a,c)\cdot\delta^2$$

note: δ^2 has units of area,

so has units $\frac{\text{Probability}}{\text{area}}$, i.e.

mass per unit area.

Recall that if $A \subseteq \mathbb{R}$, we have

$$P(X \in A) = \int_A f_X(x)\,dx$$

But also

$$P(X \in A) = P(X \in A, Y \in \mathbb{R})$$

$$= \iint_{A \times \mathbb{R}} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_A \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

Therefore the marginal PDF is

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy}$$

Similarly

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx}$$

Ex. Joint Uniform continuous r.v. on $R = [a,b] \times [c,d]$,

$$f_{X,Y}(x,y) = \begin{cases} \alpha & \text{if } (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

We must have

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \iint_R \alpha\,dx\,dy$$

$$= \alpha \cdot \text{area}(R)$$

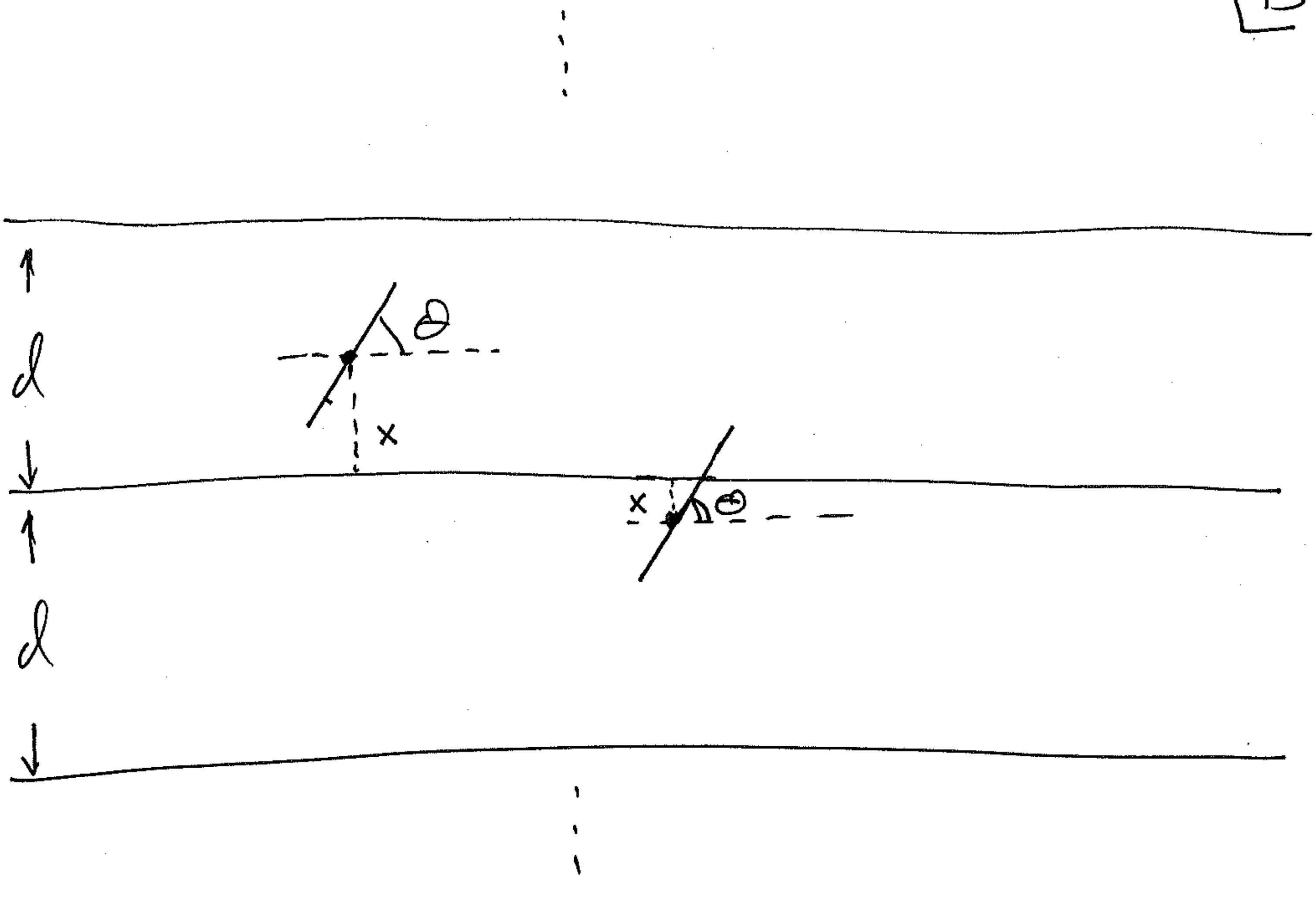
$$\therefore \quad \alpha = \frac{1}{\text{area}(R)} = \frac{1}{(b-a)(d-c)}$$

Ex Buffon's needle

A surface is ruled with parallel lines d apart. a needle of length l is tossed on the surface. what is Prob. that needle intersects one of the lines.

Assume $l < d$, so needle intersects at most 1 line.

Let X be vertical dist from midpoint of needle to nearest line.

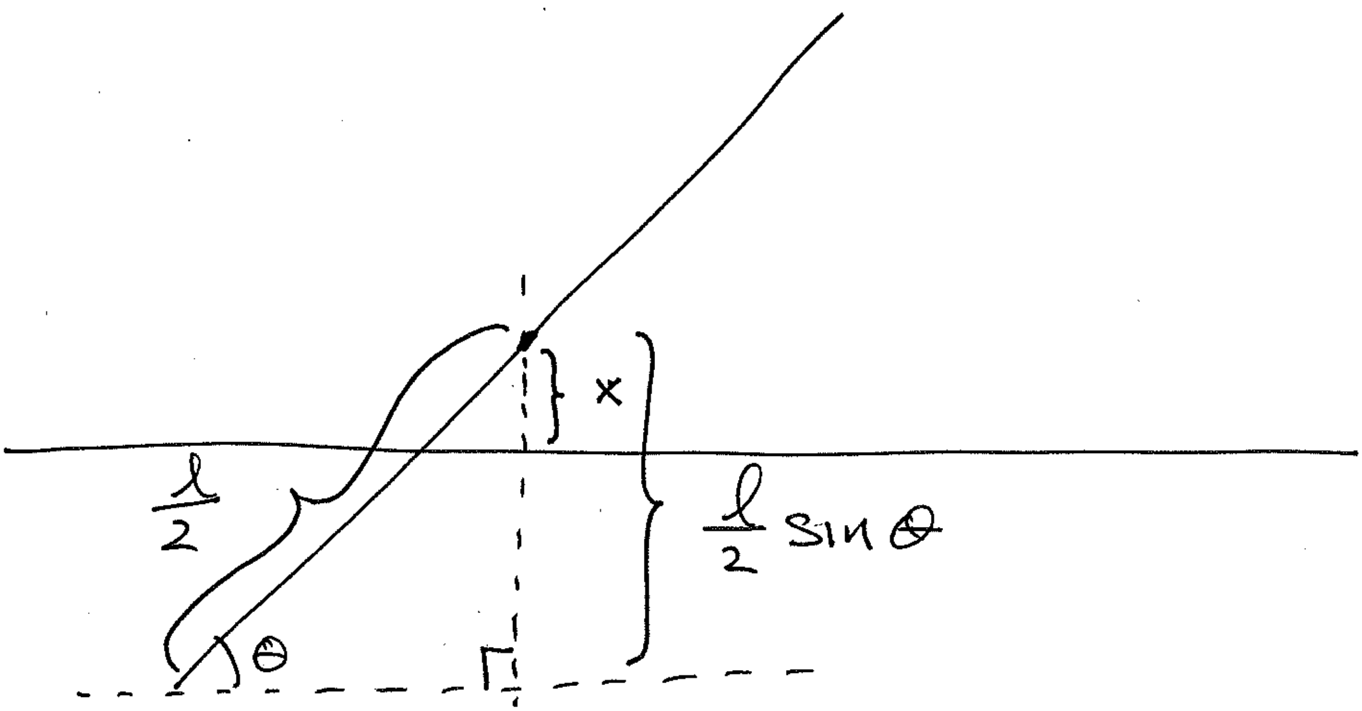


Let Θ be the acute angle made by the needle with horizontal

we model (X, Θ) as a joint uniform PDF on $[0, \frac{d}{2}] \times [0, \frac{\pi}{2}]$

so

$$f_{X,\Theta}(x,\theta) = \begin{cases} \frac{4}{\pi d} & 0 \le x \le \frac{d}{2},\ 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$



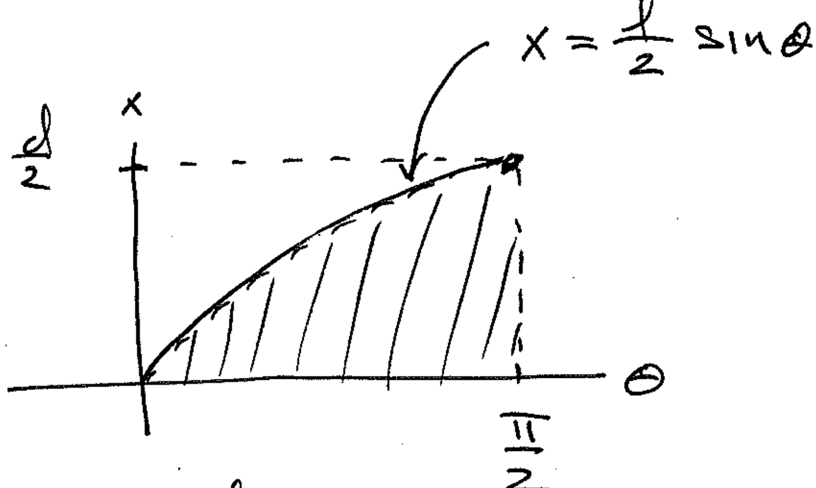
so needle intersects iff

$$\boxed{X \le \frac{l}{2}\sin\Theta}$$

Thus

$$P\left(X \le \frac{l}{2}\sin(\Theta)\right)$$

$$= \iint\limits_{x \le \frac{l}{2}\sin\theta} f_{X,\Theta}(x,\theta)\, dx\, d\theta$$



$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{\frac{l}{2}\sin\theta} dx\, d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2} \frac{l}{2}\sin\theta\, d\theta$$

$$= \frac{2l}{\pi d}\left(-\cos\theta\right)\Big|_0^{\pi/2}$$

$$= -\frac{2l}{\pi d}(0-1) = \boxed{\frac{2l}{\pi d}}$$

Pick $l = d$ so $P(\text{intersect}) = \frac{2}{\pi}$

$$\therefore \quad \pi = \frac{2}{P(\text{intersect})}.$$