CSE 107 2-1-24

2.7 Independence

Defn

we say r.v.s X, Y are <u>independent</u> iff for all $x \in \text{range}(X)$ and $y \in \text{range}(Y)$ we have

$$\{X = x\} \text{ and } \{Y = y\}$$

are independent, i.e

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

i.e.

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)}$$

If $P_Y(y) > 0$ for all $y \in \text{range}(Y)$,

this is equiv. to

$$\frac{P_{X,Y}(x,y)}{P_Y(y)} = P_X(x)$$

i.e.

$$P_{X|Y}(x|y) = P_X(y)$$

We can extend this to a cond. Prob. law: $P(\cdot \mid A)$ where $P(A) > 0$.

Defn

two r.v.s X, Y are conditionally independent given A, iff

$$P(X=x, Y=y \mid A) = P(X=x \mid A) \cdot P(Y=y \mid A).$$

i.e.

$$\boxed{P_{X,Y|A}(x,y) = P_{X|A}(x) \cdot P_{Y|A}(y)}$$

<u>Theorem</u>: If X, Y are indep., then

$$\boxed{E[X \cdot Y] = E[X] \cdot E[Y]}$$

<u>Proof</u>

$$E[X \cdot Y] = \sum_x \sum_y xy\, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y xy\, p_X(x) \cdot p_Y(y)$$

$$= \left(\sum_x x\, p_X(x)\right) \cdot \left(\sum_y y\, p_Y(y)\right)$$

$$= E[X] \cdot E[Y].$$

∎

Theorem

If X, Y are indep., then so are $g(X)$ and $h(Y)$, for any functions g, h.

see Problem #44 on P. 134.

Thus

$$\boxed{E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]}$$

another consequence: if X, Y are independent, then

$$\boxed{\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)}$$