CSE 107 1-9-24

1.1 Sets: Review from CSE 16

Read this.

1.2 Probabilistic Models

A Probability Model consists 2 things

(1) A set Ω called the sample space. Elements $\omega \in \Omega$ are called outcomes (of some experiment). A subset $A \subseteq \Omega$ is an event.

(2) A <u>Probability law</u> P that assigns to an event $A \subseteq \Omega$ a real number $P(A)$ called <u>Probality of A</u>.

<u>Axioms for P</u>

(i) $P(A) \geq 0$ for all $A \subseteq \Omega$

(ii) $P(\Omega) = 1$

(iii) if $A, B \subseteq \Omega$ are disjoint (i.e. $A \cap B = \emptyset$), then

- $P(A \cup B) = P(A) + P(B)$

(<u>additivity</u>)

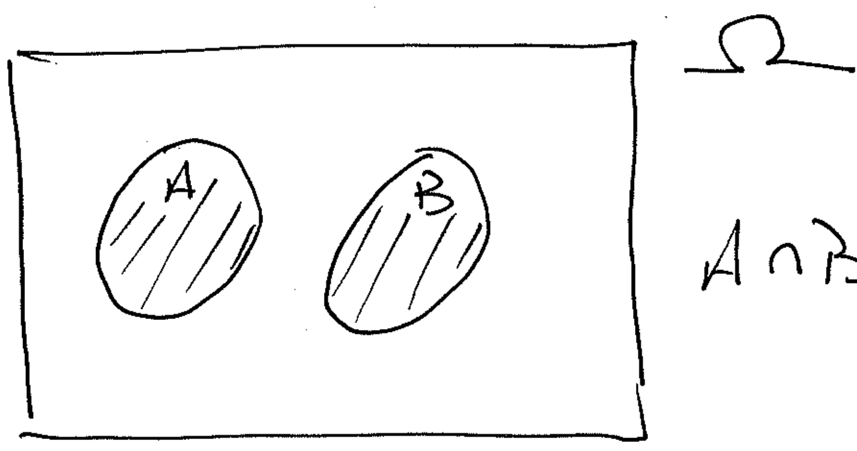
<u>more generally</u>

<u>countable additivity</u>!

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

provided $A_1, A_2, \ldots$ is pairwise disjoint: $A_i \cap A_j = \emptyset$ for all $i \neq j$

Picture!



$A \cap B = \emptyset$.

<u>Ex.</u> Flip a <u>fair</u> coin.

$$\Omega = \{H, T\}$$

<u>notation</u>

identify: H with $\{H\}$

T " $\{T\}$

Since $\Omega = \{H\} \cup \{T\}$ and

$\{H\} \cap \{T\} = \phi$, we have

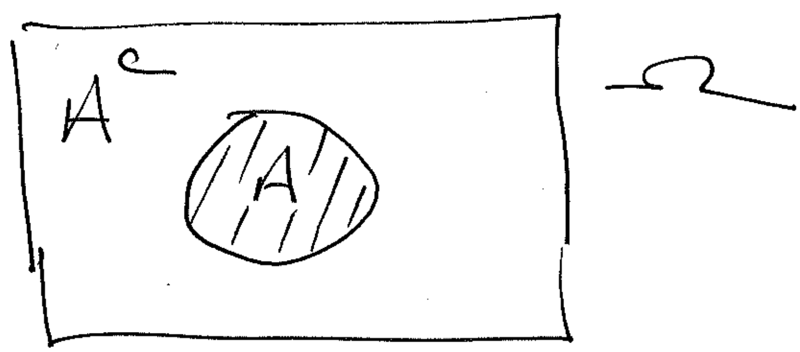
$$P(H) + P(T) = P(\{H, T\})$$
$$= P(\Omega)$$
$$= 1$$

fair means $P(H) = P(T)$.

$\therefore P(H) = \frac{1}{2} = P(T)$.

In general, if $A \subseteq \Omega$, then

$$A \cup A^c = \Omega$$



and $A \cap A^c = \emptyset$. Thus

$$P(A) + P(A^c) = P(\Omega) = 1$$

Therefore

$$0 \leq P(A) \leq 1$$

evidently $P(\cdot)$ is a function

$$P : \mathbb{P}(\Omega) \longrightarrow [0, 1]$$

letting $A = \Omega$

$$
\begin{aligned}
1 &= P(\Omega) \\
&= P(\Omega \cup \Omega^c) \\
&= P(\Omega \cup \phi) \\
&= P(\Omega) + P(\phi) \\
&= 1 + P(\phi)
\end{aligned}
$$

$\therefore \quad P(\phi) = 0$

$\square$

Going back to example

$$\Omega = \{H, T\}$$

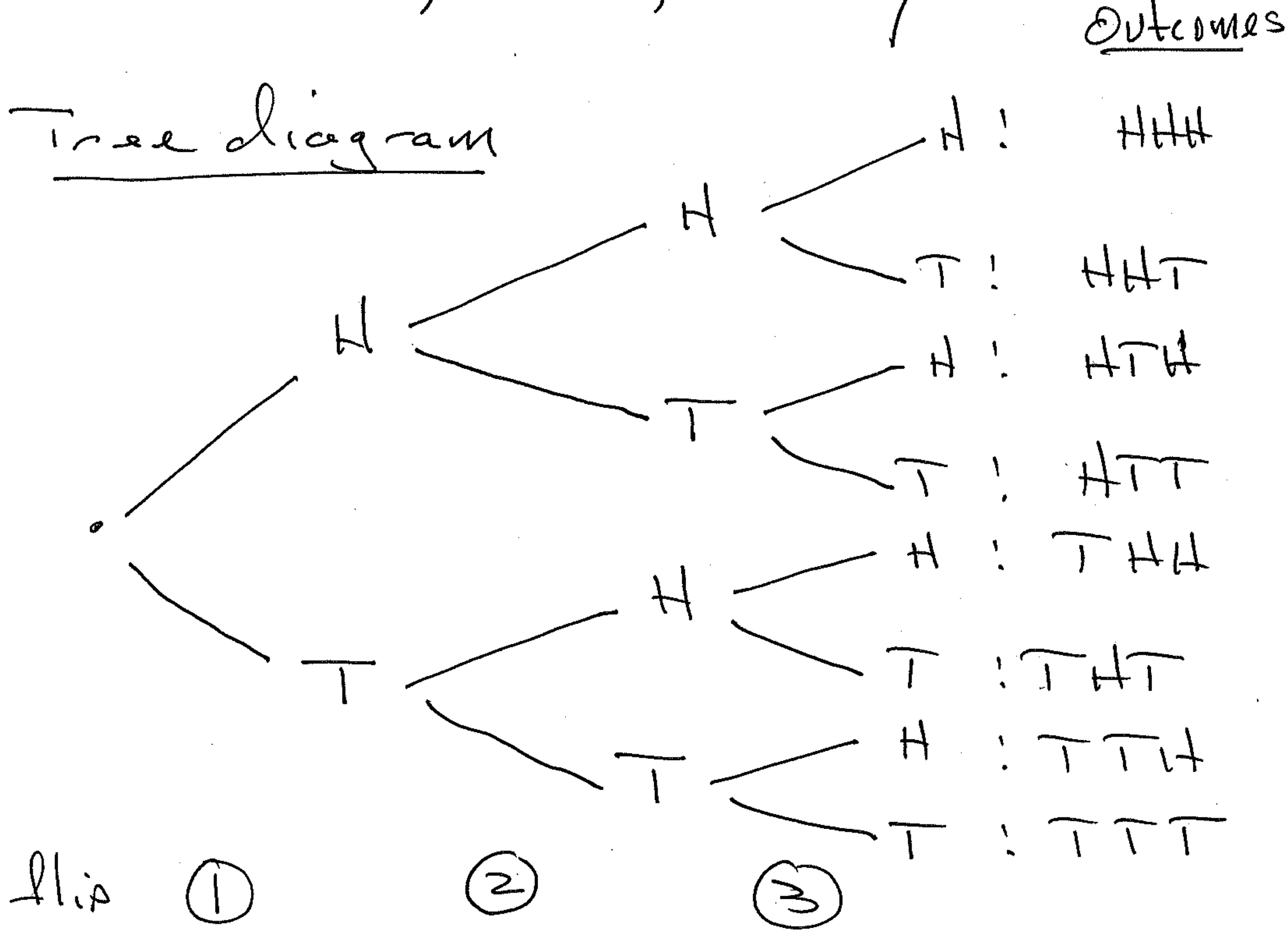
$$\mathbb{P}(\Omega) = \left\{ \Omega, \{H\}, \{T\}, \phi \right\}$$

with Probabilities: $1, \frac{1}{2}, \frac{1}{2}, 0$.

Ex. flip 3 fair coins

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Tree diagram



'fair' means each outcome is equally likely:

$$P(HHH) = \ldots\ldots = P(TTT) = \frac{1}{8}$$

so

$$A = \{\text{exactly 1 head}\}$$

$$= \{HTT, THT, TTH\}$$

$$= \{HTT\} \cup \{THT\} \cup \{TTH\}$$

$$\therefore P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Generalize:

Discrete Probability law

If $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, then

$P(\cdot)$ is specified by singleton

events

$$P(\omega_i) = P(\{\omega_i\}) \text{ for } 1 \le i \le n$$

so if $A = \{\omega_2, \omega_5, \omega_{13}\}$ then

$$P(A) = P(\omega_2) + P(\omega_5) + P(\omega_{13}).$$

Special case

Discrete Uniform Probability law

If Ω is finite and all $\omega \in \Omega$ are equally likely, then

$$P(A) = \frac{|A|}{|\Omega|}$$

for all $A \subseteq \Omega$.

Ex. 2 6-sided dice are thrown. The outcomes are equally likely.

What is Ω?

sum =	2	3	4	5	6	7	
	11	12	13	14	15	16	
	21	22	23	24	25	26	8
	31	32	33	34	35	36	9
	41	42	43	44	45	46	10
	51	52	53	54	55	56	11
	61	62	63	64	65	66	12

$|\Omega| = 36$

find

$$P(\text{sum} = 2) = \frac{1}{36}$$

$$P(\text{sum} = 3) = \frac{2}{36}$$

$$P(\text{sum} = 4) = \frac{3}{36}$$

$\vdots$

$$P(\text{sum is even}) \stackrel{?}{=} \frac{18}{36} = \frac{1}{2}$$

$\vdots$

$$P(\text{at least one } 3) = \frac{11}{36}$$

Continuous models

Next time . . .