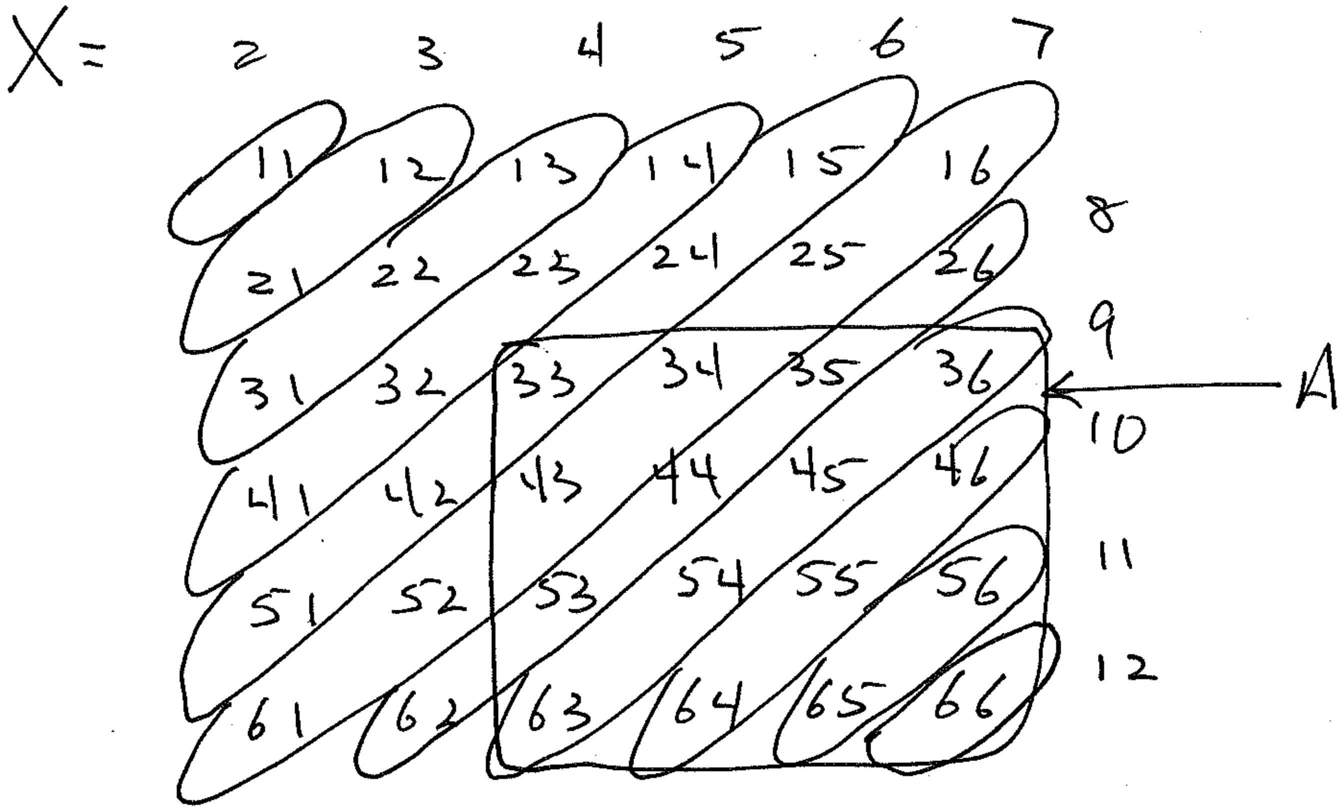
CSE 107 1-30-24

• mid 1 on Thur Feb. 1

<u>Ex.</u>

roll 2 indep. fair 6-sided dice.
let X be the sum of faces.

$$p_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & k = 3, 11 \\ 3/36 & k = 4, 10 \\ 4/36 & k = 5, 9 \\ 5/36 & k = 6, 8 \\ 6/36 & k = 7 \\ 0 & \text{otherwise} \end{cases}$$



let $A = \{\min \geq 3\}$ $\therefore P(A) = \frac{16}{36}$

$$P_{X|A}(k) = \begin{cases} 1/16 & k = 6, 12 \\ 2/16 & k = 7, 11 \\ 3/16 & k = 8, 10 \\ 4/16 & k = 9 \\ 0 & \text{otherwise} \end{cases}$$

Ex.

flip a weighted coin with $P(H) = p$ until 1^{st} heads. let X be the # of flips. so X is Geometric with Param. p.

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$$

Let A be the event that the first head occurs within n flips, i.e. $A = \{X \le n\}$.

Then

$$P(A) = \sum_{k=1}^{n} P_X(k)$$

$$= \sum_{k=1}^{n} (1-p)^{k-1} \cdot p$$

$$= p \sum_{k=0}^{n-1} (1-p)^k$$

$$= \cancel{p} \frac{1-(1-p)^n}{\cancel{1-(1-p)}} = 1-(1-p)^n \checkmark$$

another way:

$$P(A) = 1 - P(A^c)$$

$$= 1 - P(1^{st}\ n \text{ flips tails})$$

$$= 1-(1-p)^n \checkmark$$

Thus

$$p_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1} \cdot p}{1-(1-p)^n} & \text{if } k = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

<u>Exercise</u>: check that

$$\sum_{k=1}^{n} p_{X|A}(k) = 1 .$$ ■

Now let Y be another r.v. with

$$p_Y(y) = P(Y = y) > 0$$

for all $y \in \text{range}(Y)$.

Now we specialize to events

$$A = \{ Y = y \}$$

<u>Defn</u>

The conditional PMF of X given Y is

$$P_{X|Y}(x|y) = P(X = x \mid Y = y)$$

$$= \frac{P(X=x, Y=y)}{P(Y=y)}$$

also

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

Thus

$$\boxed{P_{X,Y}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)}$$

also

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)}$$

Ex

A professor answers questions wrongly with Prob. $\frac{1}{4}$, indep. of other questions. The # of questions asked is 0, 1, 2 with equal probability $\frac{1}{3}$.

let

$$X = \#\text{questions asked}$$

and

$$Y = \#\text{questions answered wrongly}$$

we have

$$P_X(x) = \begin{cases} \frac{1}{3} & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Also

$$P_{Y|X}(y|x) = \begin{cases} & \underline{\#\text{asked}} \quad \underline{\#\text{wrong}} \\ 1 & \text{if } \; x = 0, \; y = 0 \\ \frac{3}{4} & \quad\;\; x = 1, \; y = 0 \\ \frac{1}{4} & \quad\;\; x = 1, \; y = 1 \\ \frac{9}{16} & \quad\;\; x = 2, \; y = 0 \\ \frac{3}{8} & \quad\;\; x = 2, \; y = 1 \\ \frac{1}{16} & \quad\;\; x = 2, \; y = 2 \\ 0 & \quad\;\; \text{otherwise} \end{cases}$$

Therefore $P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)$

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{3} & \text{if } x=0,\ y=0 \\ \frac{1}{4} & x=1,\ y=0 \\ \frac{1}{12} & x=1,\ y=1 \\ \frac{3}{16} & x=2,\ y=0 \\ \frac{1}{8} & x=2,\ y=1 \\ \frac{1}{48} & x=2,\ y=2 \\ 0 & \text{otherwise} \end{cases}$$

can no use joint PMF to find marginal $P_Y(y)$.

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

Thus

$$p_Y(y) = \begin{cases} 37/48 & \text{if } y=0 \\ 10/48 & y=1 \\ 1/48 & y=2 \\ 0 & \text{otherwise} \end{cases}$$

Exercise!

find cond. PMF $p_{X|Y}(x|y)$.

see summary! p. 103

Conditional expectation

Let X be a r.v. and $A \subseteq \Omega$ with $P(A) > 0$. Define

$$\boxed{E[X|A] = \sum_x x \cdot P_{X|A}(x)}$$

also

$$\boxed{E[g(X)|A] = \sum_x g(x) P_{X|A}(x)}$$

we can condition on

$$A = \{Y = y\}$$

Provided $P(Y=y) > 0$.

$$\boxed{E[X \mid Y=y] = \sum_x x \, P_{X|Y}(x|y)}$$

Also if $A_1, A_2, \dots, A_n$ form a Partition of Ω, then

<u>total expectation theorem</u>

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]}$$

Proof

By the total Prob. thm.

$$P(X=x)=\sum_{i=1}^{n} P(A_i)\cdot P(X=x \mid A_i)$$

Thus

$$p_X(x)=\sum_{i=1}^{n} P(A_i)\cdot p_{X|A_i}(x)$$

multiply by x and sum over $x \in \text{range}(X)$.

$$E[X]=\sum_x x\, p_X(x)$$

$$=\sum_x x\cdot\sum_{i=1}^{n} P(A_i)\cdot p_{X|A_i}(x)$$

$$= \sum_{x} \sum_{i=1}^{n} P(A_i) \cdot x \cdot p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \sum_{x} x \, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$$

See other identities! P. 104.

Mean & Variance of Geometric

Recall

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

must find

$$E[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$$

and

$$E[X^2] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} \cdot p$$

then

$$Var(X) = E[X^2] - E[X]^2$$

Let

$A_1 = \{X = 1\} = \{1^{st} \text{ flip is head}\}$

$A_2 = \{X > 1\} = \{1^{st} \text{ flip is tails}\}$

Then $P(A_1) = p$, $P(A_2) = 1-p$

<u>Note</u>: $E[X|A_1] = 1$, and

$E[X|A_2] = E[1+X] = 1 + E[X]$

Thus (by total expect. thm)

$E[X] = p \cdot 1 + (1-p)(1+E[X])$

$\cancel{E[X]} = \cancel{p} + 1 - \cancel{p} + \cancel{E[X]} - pE[X]$

Thus

$$p\,E[X] = 1$$

$$\therefore \quad \boxed{E[X] = \frac{1}{p}}$$

Similarly

$$E[X^2 \mid A_1] = 1$$

and

$$E[X^2 \mid A_2] = E[(1+X)^2]$$

$$= E[1 + 2X + X^2]$$

$$= 1 + 2E[X] + E[X^2]$$

$$\therefore \quad E[X^2] = p \cdot 1 + (1-p)\left(1 + 2E[X] + E[X^2]\right)$$

⋮
exercise
⋮

$$E[X^2] = \frac{2}{p^2} - \frac{1}{p}$$

therefore

$$\boxed{Var(X) = \frac{1-p}{p^2}}$$