CSE 107 1-26-24

Supplemental Lecture

<u>Ex</u>

Suppose $\text{range}(X) = \{1, 2, 3, 4\}$, and $\text{range}(Y) = \{1, 2, 3\}$. Let the joint PMF be

$p_X(x)$

	$4/20$	$5/20$	$6/20$	$5/20$	
	↑	↑	↑	↑	$p_Y(y)$
$y = 3$	$1/20$	$1/20$	$2/20$	$1/20$	→ $5/20$
$y = 2$	$2/20$	$3/20$	$3/20$	$1/20$	→ $9/20$
$y = 1$	$1/20$	$1/20$	$1/20$	$3/20$	→ $6/20$
	$x = 1$	$x = 2$	$x = 3$	$x = 4$	

$p_{X,Y}(x,y)$

note: $\{X=1\} \cup \{X=2\} \cup \{X=3\} \cup \{X=4\} = \Omega$

is a partition of Ω. consider the event $\{Y=3\}$. Then

$$\{Y=3\} = \{X=1, Y=3\} \cup \{X=2, Y=3\} \cup \{X=3, Y=3\} \cup \{X=4, Y=3\}$$

is a disjoint union. Hence

$$P(Y=3) = P(X=1, Y=3) + P(X=2, Y=3) + P(X=3, Y=3) + P(X=4, Y=3)$$

Thus

$$P_Y(3) = \sum_x P_{X,Y}(x, 3)$$

In general

$$\boxed{P_Y(y) = \sum_x P_{X,Y}(x, y)}$$ marginal PMF

Also

$$\boxed{p_X(x) = \sum_y p_{X,Y}(x,y)}$$ marginal PMF

<u>Functions of multiple r.v.s</u>

let $Z = g(X,Y)$ where

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Then

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} p_{X,Y}(x,y)$$

The expected value rule can be extended to

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y)}$$

Exercise: Prove this.

Ex. Show that

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Proof.

$$E[aX+bY+c] = \sum_x \sum_y (ax+by+c)\, p_{X,Y}(x,y)$$

$$= a\sum_x \sum_y x\, p_{X,Y}(x,y) + b\sum_y \sum_x y \cdot p_{X,Y}(x,y)$$

$$+ c\sum_x \sum_y p_{X,Y}(x,y)$$

$$= a\sum_x x\, p_X(x) + b \cdot \sum_y y\, p_Y(y) + c$$

$$= aE[X] + bE[Y] + c \quad \blacksquare$$

Ex. (Previous)

$$E[X+3Y] = E[X] + 3E[Y]$$

$$= \left(1\cdot\frac{4}{20} + 2\cdot\frac{5}{20} + 3\cdot\frac{6}{20} + 4\cdot\frac{5}{20}\right) + 3\left(1\cdot\frac{6}{20} + 2\cdot\frac{9}{20} + 3\cdot\frac{5}{20}\right)$$

$$= \cdots = \boxed{\frac{169}{20}}$$

Everything we did for 2 r.v.s can be done for 3, 4, ..., n r.v.s

Let X, Y, Z be r.v.s on Ω.

Their joint PMF is

$$P_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$$

we have 3 2-variable marginal PMFs

$$P_{X,Y}(x,y) = \sum_z P_{X,Y,Z}(x,y,z)$$

$$P_{X,Z}(x,z) = \sum_y P_{X,Y,Z}(x,y,z)$$

$$P_{Y,Z}(y,z) = \sum_x P_{X,Y,Z}(x,y,z)$$

also

$$P_X(x) = \sum_y \sum_z P_{X,Y,Z}(x,y,z)$$

same for $P_Y(y)$ and $P_Z(z)$

suppose we have $X_1, X_2, \ldots, X_n$.

Then

$$E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n + b]$$

$$= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_n E[X_n] + b$$

<u>The mean of the Binomial r.v.</u>

Let X be a Binomial r.v. with parameters n, p. Recall

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & (0 \le k \le n) \\ 0 & \text{otherwise} \end{cases}$$

Goal: compute

$$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

note that:

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i is Bernoulli with parameter p. Recall

$$E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$$

so

$$E[X] = E[X_1 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= p + \cdots + p$$

$$= n p.$$

(need this for hw 2 #7).

we'll show later that

$$Var(X) = np(1-p)$$

Ex. The hat problem

Suppose n people throw n hats into a box, then each picks a hat at random from the box.

Each assignment

$$\{hats\} \longrightarrow \{people\}$$

is equally likely. Let $X = \#$ people who get their own hat back.

Find $E[X]$.

observe: $X = X_1 + X_2 + \cdots + X_n$

where each X_i is Bernoulli
with $p = \frac{1}{n}$.

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person gets hat back} \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X_i] = p.$$

so $E[X] = np = n \cdot \frac{1}{n} = \boxed{1}$

2.6 Conditioning

Given $X : \Omega \to \mathbb{R}$ a discrete r.v., any valid probability law $P(\cdot)$ can be used to define a PMF

$$P_X(x) = P(X = x)$$

Let $A \subseteq \Omega$ be an event, with $P(A) > 0$.

Defn the conditional PMF of X given A is

$$\boxed{P_{X|A}(x) = P(\{X=x\} \mid A) = \frac{P(\{X=x\} \cap A)}{P(A)}}$$

<u>note</u>: the events $\{X=x\}\cap A$ form a partition of A:

$$A = \bigcup_x \left(\{X=x\}\cap A\right)$$

so by total prob. thm

$$P(A) = \sum_x P\left(\{X=x\}\cap A\right).$$

Thus

$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X=x\}\cap A)}{P(A)}$$

$$= \frac{1}{P(A)} \cdot \sum_x P\left(\{X=x\}\cap A\right)$$

$$= \frac{1}{\cancel{P(A)}} \cdot \cancel{P(A)} = 1.$$

showing $P_{X|A}(x)$ is a valid PMF.