cse 107 1-25-24

- midterm 1: Thur. 2-1-24
- Remarks on lab 2.

suppose $a=3$, $c=5$

$\boxed{\begin{matrix} \cancel{A} & \cancel{A} & \cancel{A} \\ \cancel{C} & \cancel{C} & \cancel{C} & \cancel{C} & \cancel{C} \end{matrix}}$

<u>ch1</u>: $\cancel{C}$ $\cancel{C}$ A ← replace

<u>ch2</u>: $\cancel{C}$ A ← replace

<u>ch3</u>: $\cancel{A}$ $\cancel{A}$ C ← replace

<u>ch4</u>: $\cancel{A}$ C ← replace

<u>ch5</u>: $\cancel{C}$ $\cancel{C}$ ← last discarded is C.

Recall $E[aX+b] = aE[X]+b$

Ex. find $Var(aX+b)$

let $Y = aX+b$. will use

$$Var(Y) = E[Y^2] - E[Y]^2$$

2nd moment:

$$E[Y^2] = E[(aX+b)^2]$$

$$= \sum_x (ax+b)^2 p_X(x)$$

$$= \sum_x (a^2x^2 + 2abx + b^2) p_X(x)$$

$$= a^2 \sum_x x^2 p_X(x) + 2ab \sum_x x p_X(x) + b^2 \sum_x p_X(x)$$

$$= a^2 E[X^2] + 2ab\, E[X] + b^2$$

so

$$Var(Y) = \left(a^2 E[X^2] + 2ab E[X] + b^2\right)$$

$$- \left(a E[X] + b\right)^2$$

$$=$$

$$\vdots \quad \text{(exercise)}$$

$$= a^2 \left(E[X^2] - E[X]^2\right)$$

$$= a^2 \, Var(X)$$

Summary:

- $E[aX+b] = aE[X]+b$
- $Var(aX+b) = a^2 Var(X)$

mean and Var of Known r.v.s

Ex. Bernoulli r.v. with Parameter p.

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

so

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = \boxed{p}$$

$$E[X^2] = 1^2 \cdot p + 0^2 (1-p) = p$$

$$Var(X) = E[X^2] - E[X]^2$$
$$= p - p^2 = \boxed{p(1-p)}$$

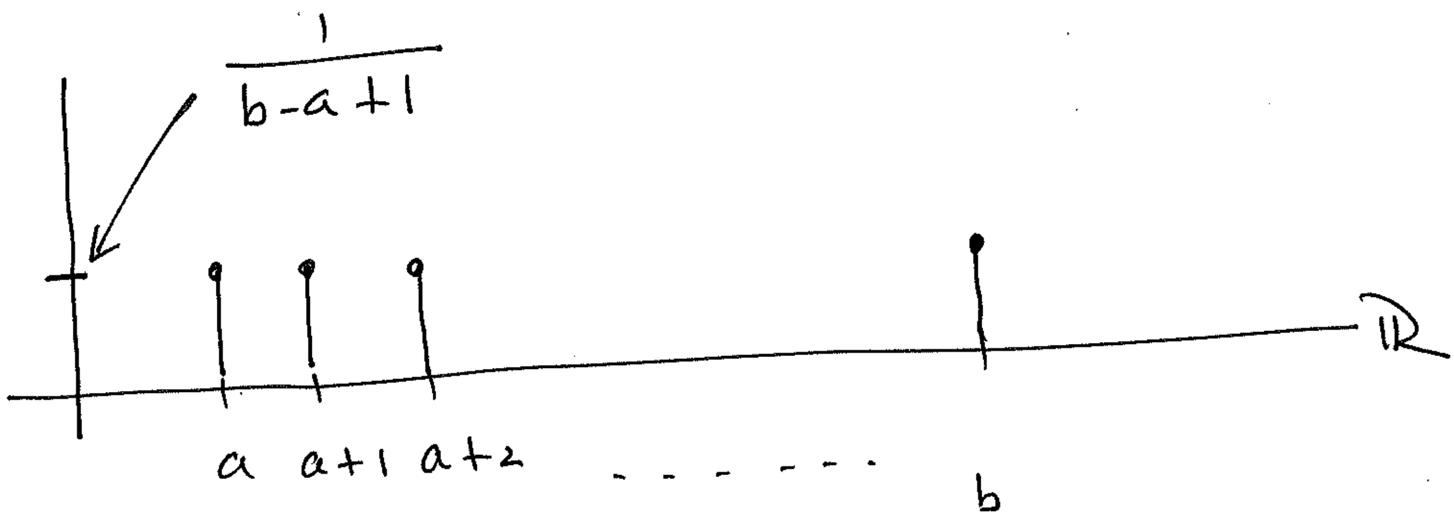
<u>Ex</u> Discrete uniform r.v.

let $a, b \in \mathbb{Z}$, suppose X distributes 'mass' uniformly among

$$\{a, \ldots, b\}$$

where $a \le b$.

i.e. all int. k with $a \le k \le b$



Thus

$$P_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } a \le k \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{k=a}^{b} k \cdot \left(\frac{1}{b-a+1}\right)$$

$$= \left(\frac{1}{b-a+1}\right) \cdot \sum_{k=a}^{b} k$$

$$= \left(\frac{1}{b-a+1}\right)\left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k\right)$$

$$= \frac{1}{b-a+1}\left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2}\right)$$

$\leftarrow$ (exercise)

$$= \frac{a+b}{2} \cdot \left(\frac{1}{b-\cancel{a}+1}\right) \cdot (b-\cancel{a}+1)$$

$$= \frac{a+b}{2}$$

To compute $Var(X)$, 1st set

$$Y = X + (1-a)$$

so $Var(Y) = Var(X + (1-a)) = Var(X)$

let $n = b - a + 1$, and observe that Y is discrete uniform on $\{1, 2, \ldots, n\}$, i.e $1 \le k \le n$, since

$$X = a \Rightarrow Y = a + (1-a) = 1$$

$$X = b \Rightarrow Y = b + (1-a) = b-a+1 = n$$

so $E[Y] = \frac{1+n}{2}$, and

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}.$$

Thus

$$Var(X) = Var(Y)$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \text{--- exercise}$$

$$= \frac{(n+1)(n-1)}{12}$$

$$= \frac{(b-a)(b-a+2)}{12}$$

Ex. Poisson r.v., Parameter $\lambda > 0$.

$$P_X(k) = \begin{cases} e^{-\lambda} \cdot \dfrac{\lambda^k}{k!} & k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

we'll show

(1) $E[X] = \lambda$ ✓

(2) $E[X^2] = \lambda^2 + \lambda$ ✓

hence

(3) $\text{Var}(X) = (\lambda^2 + \lambda) - \lambda^2 = \lambda$

Proof of (1)

$$E[X] = \sum_{k=\cancel{0}1}^{\infty} k \cdot P_X(k)$$

$$= \sum_{k=\cancel{0}1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \qquad (*)$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{k!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \cdot \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \qquad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = \lambda \qquad \blacksquare$$

Proof of (2)

$$E[X^2] = \sum_{k=\cancel{0}\,1}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$= \ldots$ exercise $\ldots$

use $k \leftarrow k+1$

use $*$

$$= \lambda(\lambda+1) = \lambda^2 + \lambda$$

$\square$

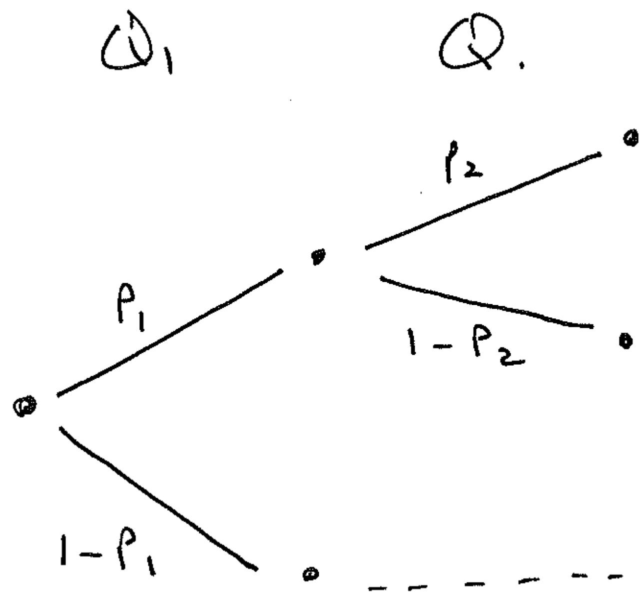
<u>Ex.</u> the Quiz Problem

A game consists of 2 questions

	$\underline{Q_1}$	$\underline{Q_2}$
P(correct)	P_1	P_2
Payoff \$	V_1	V_2

1^{st} wrong answer terminates the game. you can choose either order! (Q_1, Q_2) or (Q_2, Q_1)

- find PMF of total Payoff X. in the case (Q_1, Q_2)
- find $E[X]$.



Prob.	Payoff
$p_1 p_2$	$v_1 + v_2$
$p_1(1-p_2)$	v_1
$1-p_1$	0

$$P_X(x) = \begin{cases} p_1 p_2 & \text{if } x = v_1 + v_2 \\ p_1(1-p_2) & \text{if } x = v_1 \\ 1-p_1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = (v_1+v_2)p_1 p_2 + v_1 \cdot p_1(1-p_2) + \cancel{0\cdot(1-p_1)}$$

$$= v_1 p_1 + v_2 p_1 p_2$$

- find PMF of Y, total payoff in case (Q_2, Q_1), and $E[Y]$

$$P_Y(y) = \begin{cases} P_1 P_2 & \text{if } y = v_1 + v_2 \\ P_2(1-P_1) & \text{if } y = v_2 \\ 1-P_2 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E[Y] = v_2 P_2 + v_1 P_1 P_2$$

- Which order should we pick? we pick (Q_1, Q_2) over (Q_2, Q_1) if and only if

$$V_1 P_1 + V_2 P_1 P_2 > V_2 P_2 + V_1 P_1 P_2$$

$$V_1 P_1 - V_1 P_1 P_2 > V_2 P_2 - V_2 P_1 P_2$$

$$V_1 P_1 (1 - P_2) > V_2 P_2 (1 - P_1)$$

$$\boxed{\frac{V_1 P_1}{1 - P_1} > \frac{V_2 P_2}{1 - P_2}}$$

Exercise

analyze this problem with 3 questions.

2.5 Joint PMFs, multiple r.v.s

Defn

Given two r.v.s X, Y (with same domain Ω), their **joint PMF** is

$$p_{X,Y}(x,y) = \underbrace{P(X=x,\ Y=y)}_{\text{shorthand for}}$$

$$P(\{X=x\} \cap \{Y=y\})$$

The joint PMF can be used to find Prob. of events definable in terms of X and Y.

i.e. if $A \subseteq \text{range}(X) \times \text{range}(Y)$,

then

$$P\big((x,y) \in A\big) = \sum_{(x,y)\in A} p_{X,Y}(x,y)$$

⋮