2.3 Functions of a Random Variable

Given $g : \mathbb{R} \to \mathbb{R}$ and a r.v.

$X : \Omega \to \mathbb{R}$, we can compose

$$\begin{tikzcd} \Omega \arrow[r, "X"] \arrow[rr, bend left, "g \circ X = g(X)"] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

Then $g(X(\omega)) = g \circ X(\omega)$ defines another random variable:

$$Y = g \circ X = g(X)$$

Ex. Suppose X has PMF

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

let $Y = X^2$. what is $P_Y(y)$?

$$P_Y(y) = P(Y = y)$$

$$= P(X^2 = y)$$

$$= P(X = \pm\sqrt{y})$$

$$= P(X = \sqrt{y} \text{ or } X = -\sqrt{y})$$

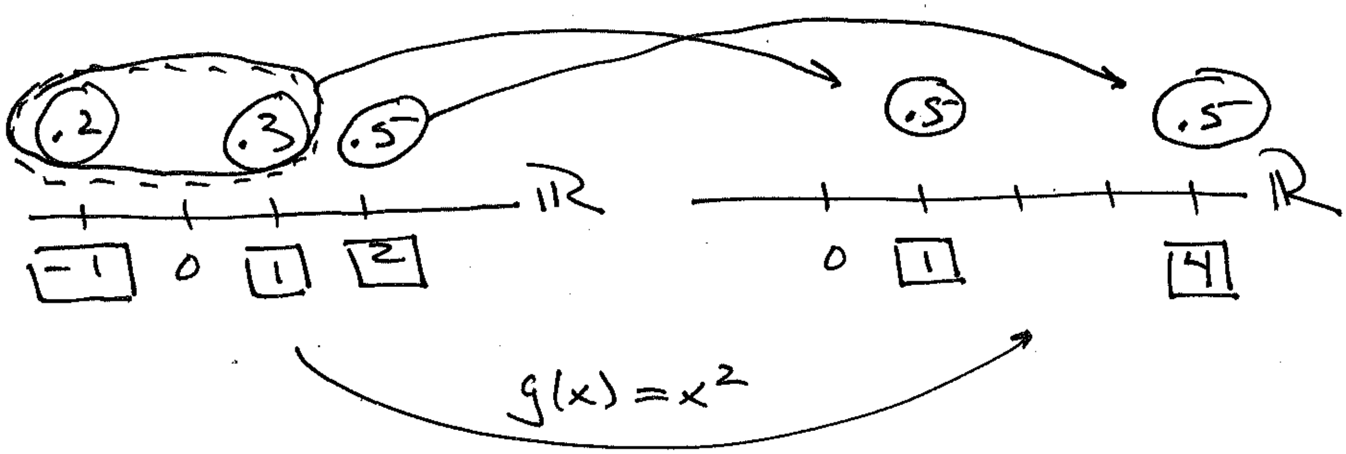
$$= P(\{X = \sqrt{y}\} \cup \{X = -\sqrt{y}\})$$

$$= P(X=\sqrt{y}) + P(X=-\sqrt{y})$$

$$= \left\{\begin{array}{lr} \cancel{.2 \text{ if } \sqrt{y}=-1} \\ .3 & \sqrt{y}=1 \\ .5 & \sqrt{y}=2 \end{array}\right\} + \left\{\begin{array}{lr} .2 \text{ if } & -\sqrt{y}=-1 \\ \cancel{.3} & \cancel{-\sqrt{y}=1} \\ \cancel{.5} & \cancel{-\sqrt{y}=2} \end{array}\right\}$$

$$= \left\{\begin{array}{ll} .3 \text{ if } & y=1 \\ .5 & y=4 \\ 0 & \text{otherwise} \end{array}\right\} + \left\{\begin{array}{ll} .2 \text{ if } & y=1 \\ 0 & \text{otherwise} \end{array}\right\}$$

$$= \begin{cases} .5 & \text{if } y=1 \\ .5 & \text{if } y=4 \\ 0 & \text{otherwise} \end{cases}$$



2.4 Expectation, Mean & Variance

Defn

The *Expected Value* of a r.v. X is

$$\boxed{E[X] = \sum_{x} x \cdot P_X(x)}$$

sum over $x \in \text{range}(X)$

also: *Expectation* or *mean*

Ex. Previous

$$E[X] = (-1)(.2) + 1\cdot(.3) + 2\cdot(.5) = \boxed{1.1}$$

$$E[X^2] = 1(.5) + 4\cdot(.5) = \boxed{2.5}$$

note: $E[X^2] \neq E[X]^2$

Expected Value rule

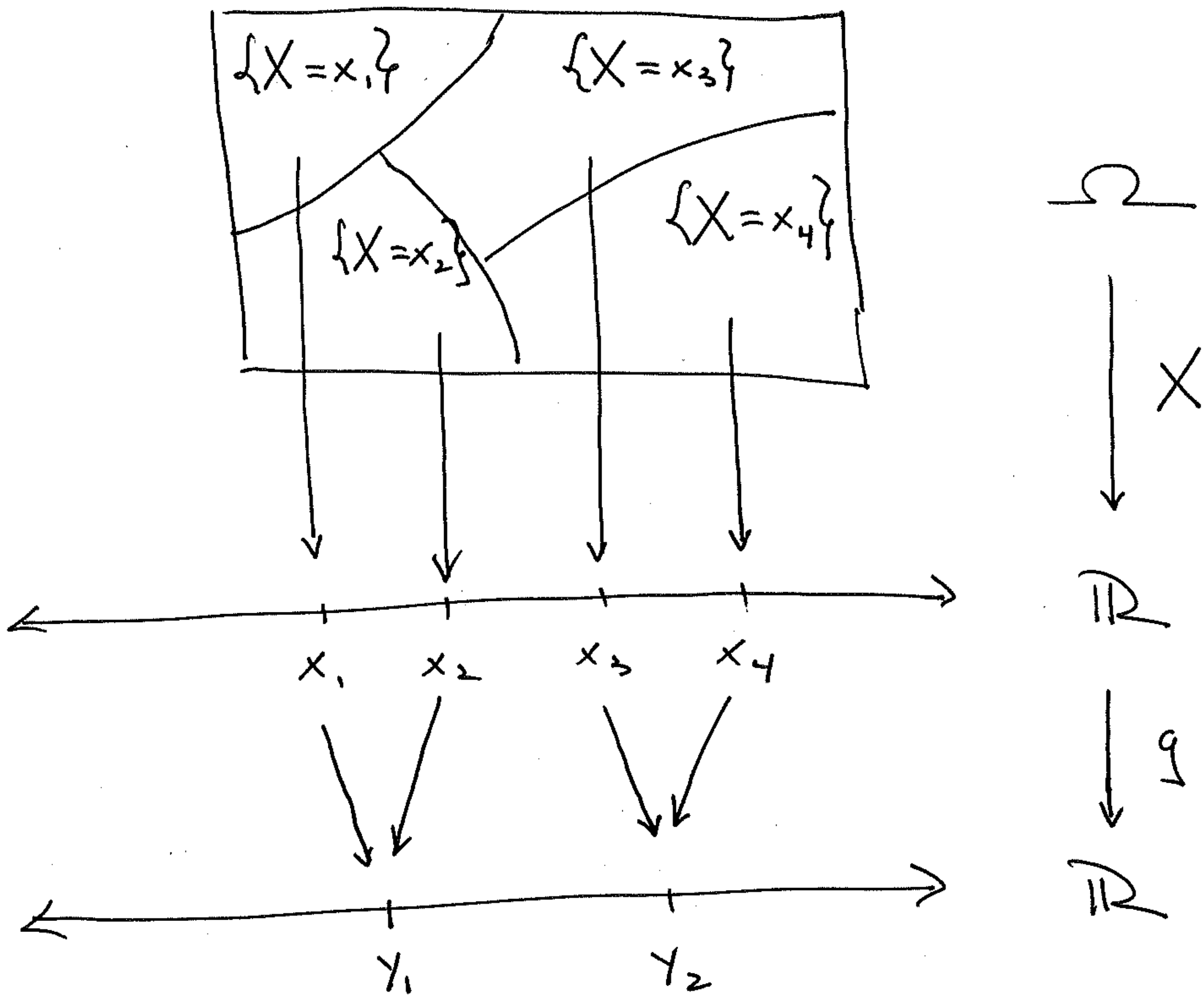
$$\boxed{E[g(X)] = \sum_x g(x) \cdot p_X(x)}$$

Ex. same ...

$$E[X^2] = (-1)^2 \cdot (.2) + (1)^2 (.3) + (2)^2 \cdot (.5)$$

$$= \ldots = \boxed{2.5}$$

Picture:



let $Y = g(X)$, then

$$E[Y] = y_1 P_Y(y_1) + y_2 P_Y(y_2)$$

$$= y_1\left(P_X(x_1)+P_X(x_2)\right) + y_2\left(P_X(x_3)+P_X(x_4)\right)$$

$$= y_1 P_X(x_1) + y_1 P_X(x_2) + y_2 P_X(x_3) + y_2 P_X(x_4)$$

$$= g(x_1)P_X(x_1) + g(x_2)P_X(x_2) + g(x_3)P_X(x_3) + g(x_4)P_X(x_4)$$

$$= \sum_x g(x)\cdot P_X(x)$$

<u>Proof</u> 1^{st} observe that

$$\{Y=y\} = \{\omega \mid Y(\omega)=y\}$$

$$= \{\omega \mid g(X(\omega))=y\}$$

$$= \bigcup_{\{x \mid g(x)=y\}} \{\omega \mid X(\omega)=x\} \quad \begin{bmatrix}\text{Disjoint}\\ \text{union}\end{bmatrix}$$

Thus

$$p_Y(y) = P(Y=y)$$

$$= \sum_{\{x \mid g(x)=y\}} P(X=x)$$

now

$$E[Y] = \sum_y y\, p_Y(y)$$

$$= \sum_y y \cdot \sum_{\{x \mid g(x)=y\}} p_X(x)$$

$$= \sum_y \sum_{\{x \mid g(x)=y\}} y\, p_X(x)$$

$$= \sum_y \sum_{\{x \mid g(x)=y\}} g(x)\, p_X(x)$$

$$= \sum_x g(x) p_X(x).$$

Hence

$$E[g(X)] = \sum_x g(x) p_X(x).$$

$\blacksquare$

<u>Ex</u> $Y = aX + b$ $(a \neq 0)$. Then

$$E[aX+b] = \sum_x (ax+b) p_X(x)$$

$$= \sum_x \left(ax p_X(x) + b p_X(x) \right)$$

$$= a \sum_x x p_X(x) + b \sum_x p_X(x)$$

$$= a E[X] + b$$

Thus: $E[aX+b] = aE[X] + b$

in Particular:

if $b=0$: $E[aX] = aE[X]$

if $a=0$: $E[b] = b$

<u>Defn</u>

The n^{th} moment of X $(n = 0, 1, 2, \ldots)$

is

$$E[X^n]$$

so $E[X]$ is 1^{st} moment, and

$E[X^2]$ is 2^{nd} moment.

<u>Defn</u>

The <u>Variance</u> of X is the mean of the r.v. $(X - E[X])^2$,

i.e.

$$\boxed{\mathrm{Var}(X) = E\left[(X - E[X])^2\right]}$$

note: $\mathrm{Var}(X) \geq 0$.

<u>Defn</u>

The <u>standard deviation</u> of X is

$$\boxed{\sigma_X = \sqrt{\mathrm{Var}(X)}}$$

Ex. (previous)

$$p_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & x = 1 \\ .5 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$E[X] = 1.1$$

$$E[X^2] = 2.5$$

By expected value rule

$$Var(X) = E[(X - E[X])^2]$$

$$= E[(X - (1.1))^2]$$

$$= \sum_x (x - (1.1))^2 \cdot p_X(x)$$

$$= (-1-(1.1))^2 \cdot (.2) + (1-(1.1))^2 (.3) + (2-(1.1))^2 (.5)$$

$$= \cdots = \boxed{1.29}$$

and $\sigma_X = \sqrt{1.29} = \boxed{1.1358}$

Variance in terms of moments

$$\boxed{Var(X) = E[X^2] - (E[X])^2}$$

EX (previous)

$$Var(X) = (2.5) - (1.1)^2$$

$$= \cdots\cdots = \boxed{1.29}$$

Proof

let $\mu = E[X]$. Then

$$\mathrm{Var}(X) = E[(X-\mu)^2]$$

$$= \sum_x (x-\mu)^2 \cdot p_X(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2)\, p_X(x)$$

$$= \sum_x x^2 p_X(x) - 2\mu \sum_x x\, p_X(x) + \mu^2 \cdot \sum_x p_X(x)$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - E[X]^2. \quad \blacksquare$$

Exercise

find $Var(aX+b)$.