CSE 107 1-19-24

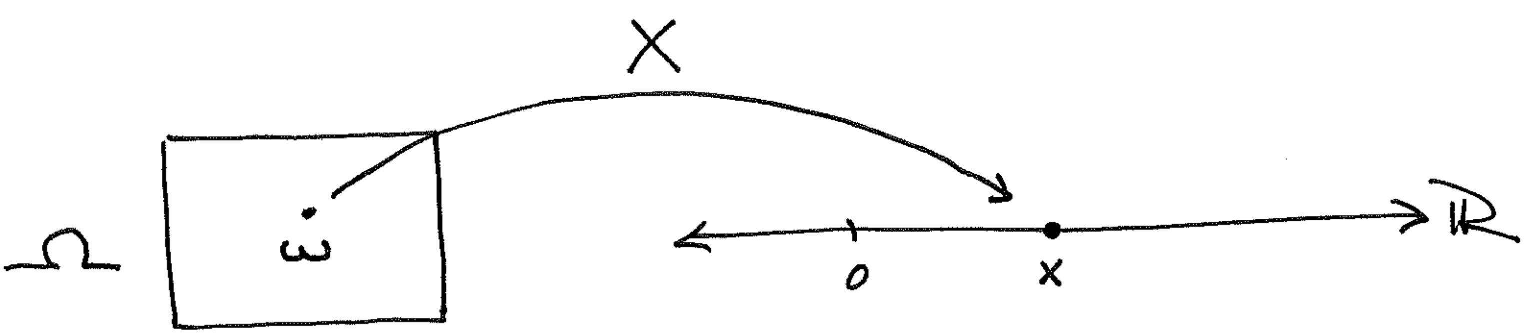
Supplemental lecture

## 2.1 Random Variables

**Defn**

A random variable is a function

$$X : \Omega \to \mathbb{R}$$



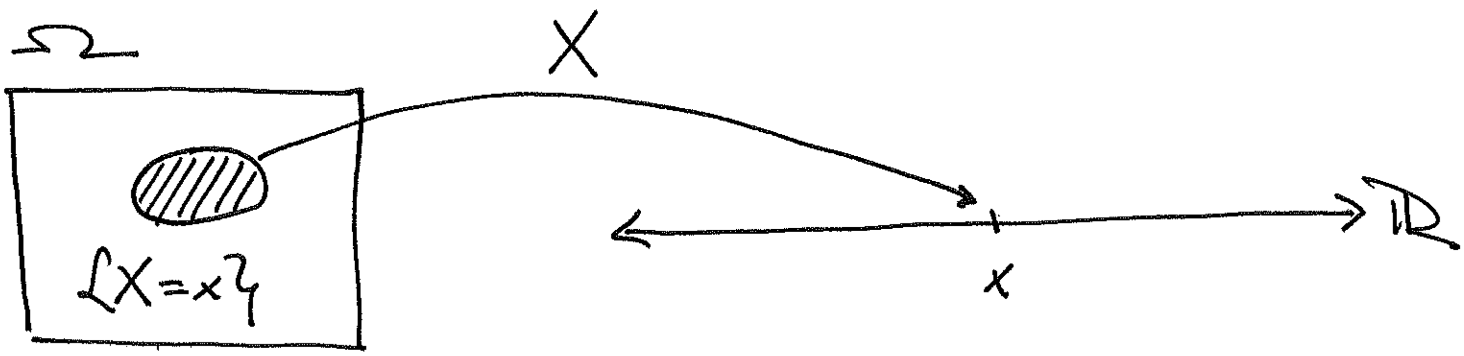

**notation**

We always use capitol letters for random variables, and lower case for values.

$$X(\omega) = x$$

<u>Also</u>: we denote by $\{X=x\}$ the event

$$\{X=x\}=\{\omega\in\Omega \mid X(\omega)=x\}\subseteq\Omega$$



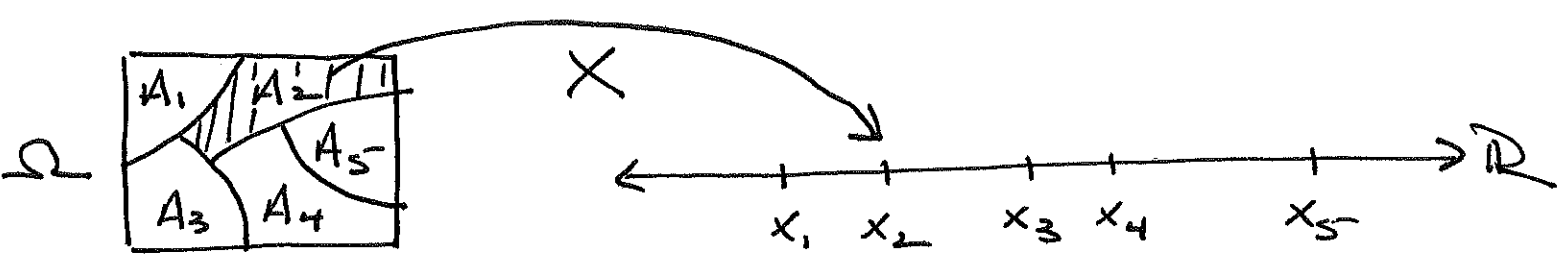

"Preimage" of $x$ under $X$.

<u>Also</u>:

$X$ is <u>discrete</u> iff its range

$$\text{range}(X)=\{X(\omega)\mid\omega\in\Omega\}\subseteq\mathbb{R}$$

is a discrete subset of $\mathbb{R}$




$$A_i=\{X=x_i\}$$

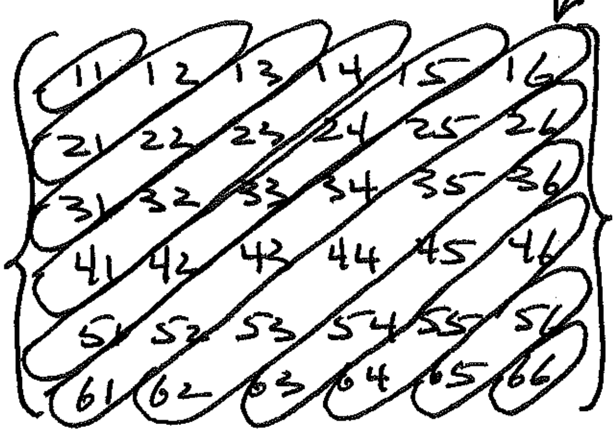
Also: $X$ is called <u>continuous</u> ..... later

Also: some r.v.s are neither discrete nor continuous.

EX. Roll 2 fair independent dice. Let

$X((i,j)) = i+j$.

$\{X=7\}$




range$(X) =$
$\{2,3,4,5,6,7,8,9,10,11,12\}$

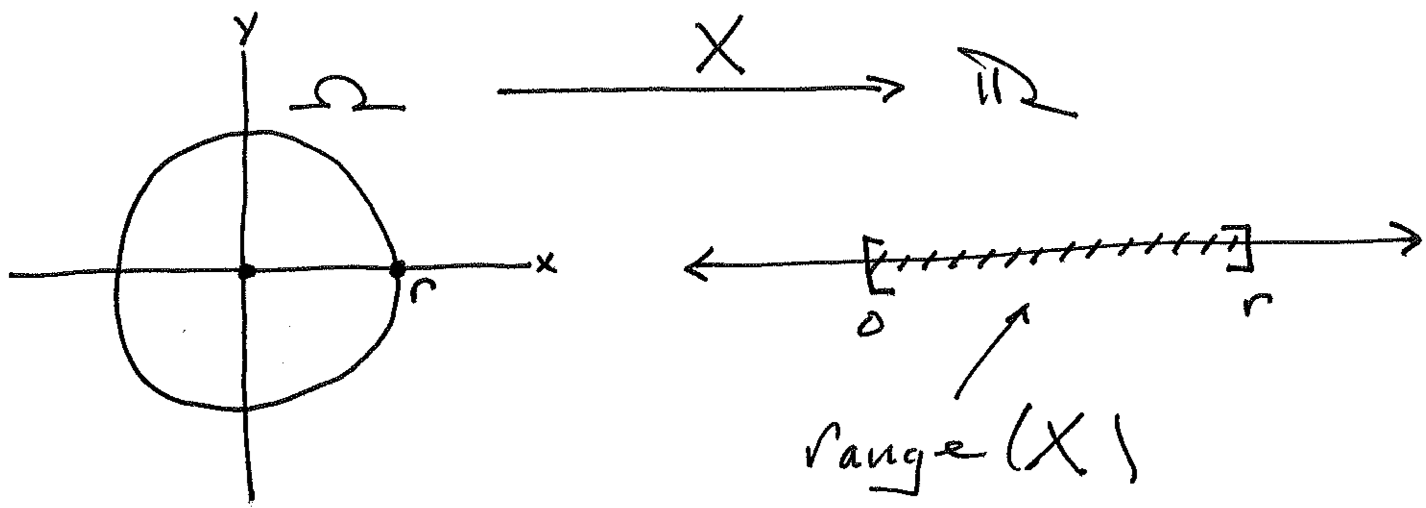
$$\Omega \xrightarrow[X]{} \mathbb{R}$$

$X$ is a discrete r.v.

Ex. Throw a dart at a circular target centered at $(0,0) \in \mathbb{R}^2$. Let

$$X((x,y)) = \sqrt{x^2+y^2}$$

the distance from dart at $(x,y)$ to $(0,0)$.




This is a continuous r.v.

## 2.2 Probability mass function

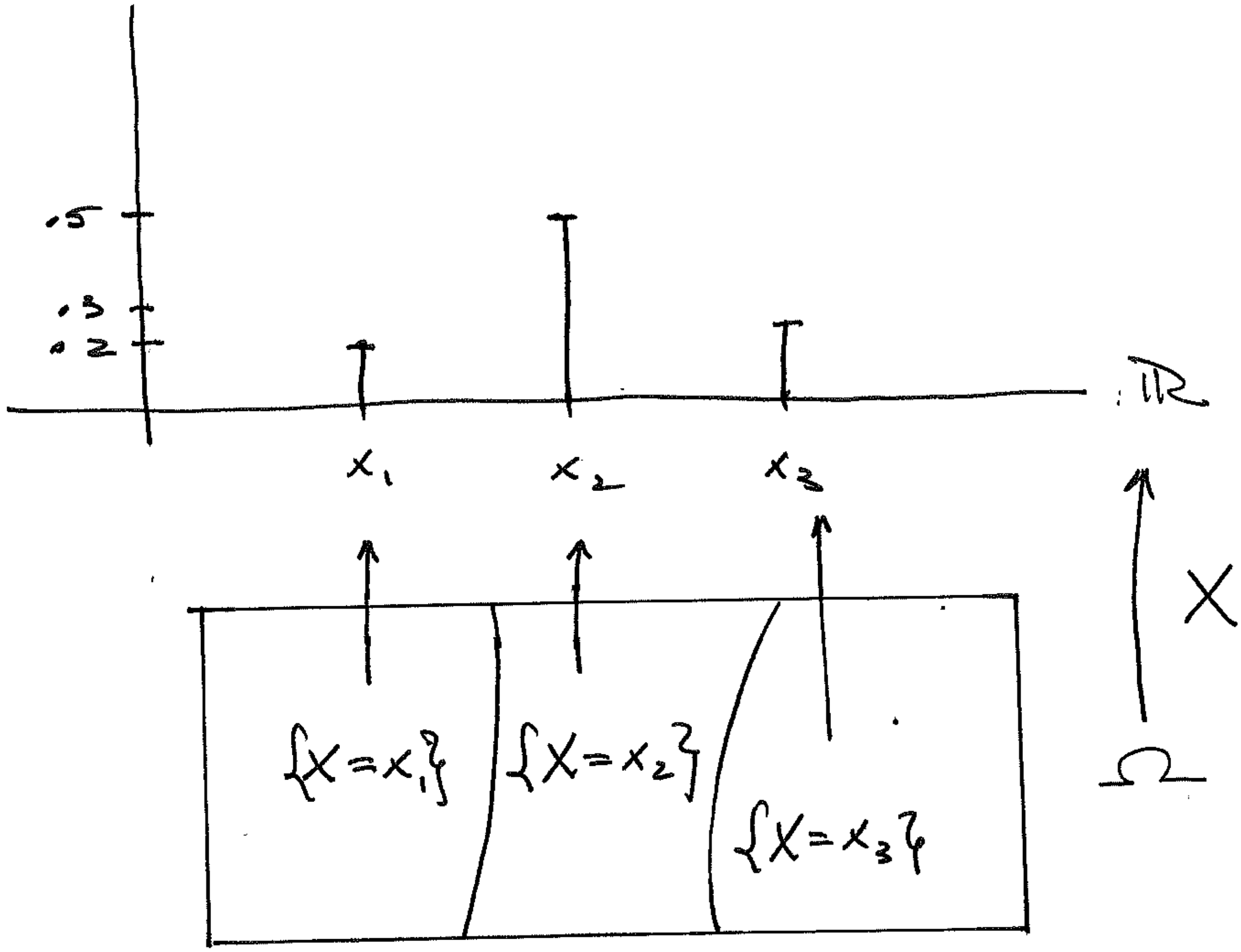
**Defn**

Let $X$ be a discrete r.v. The <u>probability mass function</u> (PMF) of $X$ is

$$p_X(x) = P(\{X = x\}) = P(X = x).$$

Observe $p_X : \mathbb{R} \to [0,1]$. Also note necessarily

$$\sum_{x \in \mathrm{range}(X)} p_X(x) = 1$$




$$\sum_{x \in \text{range}(X)} p_X(x) = (.2) + (.5) + (.3) = 1.0$$

$$p_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .5 & \text{if } x = x_2 \\ .3 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

## The Bernoulli Random Variable

Perform one Bernoulli trial, let

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is success (heads)} \\ 0 & \text{if } \omega \text{ is failure (tails)} \end{cases}$$

its PMF

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

note:
$p$ is called the parameter

where $p = P(\text{success}) = P(\text{heads})$ and therefore $1-p = P(\text{failure}) = P(\text{tails})$.

The Bernoulli r.v. represents systems with 2 states

- on, off
- working, not working

⋮

# The Binomial random variable

Perform a sequence of $n$ independent Bernoulli trials, each with parameter $p = P(\text{success})$. Let

$$X(\omega) = \#\text{successes in } n \text{ trials}.$$

The PMF of $X$ consists of Binomial probabilities

$$p_X(k) = P(X=k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$ (and 0 otherwise)

<u>check</u>: By the Binomial theorem

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+(1-p))^n = 1^n = 1 \checkmark$$

Binomial r.v. has parameters: $n, p$

# The Geometric Random Variable

Perform independent Bernoulli trials (with parameter $p$) until the 1st success (heads). Let $X$ be the # of trials performed.

$$X = \text{\# of trials until } 1^{st} \text{ success.}$$

note:

$$\text{range}(X) = \{1, 2, 3, 4, \ldots\}$$

we see $P_X(k) = P(X=k)$ is the probability of getting a seq. of $(k-1)$ failures, followed by 1 success

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Geometric r.v. has parameter : $p$

Note that

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$\left\{ \begin{array}{l} \text{replace} \\ k \text{ by } k+1 \end{array} \right.$

$$= p \cdot \sum_{k=0}^{\infty} (1-p)^{k}$$

$$= p \cdot \sum_{k=0}^{\infty} q^{k}$$

$\left\{ \begin{array}{l} \text{Replace} \\ p-1 \text{ by } q \\ \text{i.e. } q = 1-p \\ \therefore\ 1-q = p \end{array} \right.$

$$= p \cdot \left( \frac{1}{1-q} \right)$$

$$= p \cdot \frac{1}{p} = 1$$

# The Poisson Random Variable

A Poisson r.v. $X$ has PMF

$$P_X(k) = \begin{cases} e^{-\lambda} \cdot \dfrac{\lambda^k}{k!} & \text{for } k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

Poisson r.v. has parameter: $\lambda > 0$

Note: we defined $X$ by giving its PMF, not by specifying a fcn. $X : \Omega \to \mathbb{R}$.

check:

$$\sum_{k=0}^{\infty} P_X(k) = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \left( \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= e^{-\lambda} \cdot e^{\lambda} = e^{-\lambda + \lambda}$$

$$= e^{0} = 1. \quad \checkmark$$

The Poisson approximates the Binomial where $\lambda = n \cdot p$ where $n \to \infty$ and $p \to 0$ and $\lambda$ is constant.

Precisely:

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} \cdot p^k (1-p)^{n-k} \quad (k=0,1,\dots,n)$$

when $\lambda = np$, $n$ large, $p$ small.

even more precisely: let $\lambda > 0$ be constant, let $p = \frac{\lambda}{n}$:

$$\boxed{\lim_{n\to\infty} \binom{n}{k} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}}$$

Ex. let $n = 200$, $\lambda = 2$, $p = \frac{\lambda}{n} = .01$

The Prob. of $k=4$ successes in 200 independent Bernoulli trials (with .01 Prob. of success) is

$$\binom{200}{4}(.01)^4(.99)^{196} = \boxed{.0902197}$$

and this is well approx. by

$$e^{-2}\cdot\frac{2^4}{4!} = \boxed{.0902235}$$

see handout for proof of limit.