continuing last example

Define

$$C = \{\text{sum} = 5\} = \{14, 23, 32, 41\}$$

$$P(C) = \frac{4}{36} = \frac{1}{9}$$

Is C independent of A_2 ?

$$A_2 = \{21, 22, 23, 24, 25, 26\}$$

$$A_2 \cap C = \{23\}$$

$$P(A_2 \cap C) = \frac{1}{36}, \quad P(A_2) = \frac{1}{6}$$

$$P(A_2) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54} \neq \frac{1}{36}$$

$\therefore$ C, A_2 are <u>not</u> independent.

Define

$D = \{\text{doubles}\} = \{11, 22, 33, 44, 55, 66\}$

Is D independent of A_2 ?

$P(D) = \frac{6}{36} = \frac{1}{6}$

$D \cap A_2 = \{22\} \quad \therefore P(D \cap A_2) = \frac{1}{36}$

$P(D) \cdot P(A_2) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(D \cap A_2)$

$\therefore$ D, A_2 <u>are</u> independent. ■

<u>Exercise</u>

Let $A, B \subseteq \Omega$. Prove that t.f.a.e

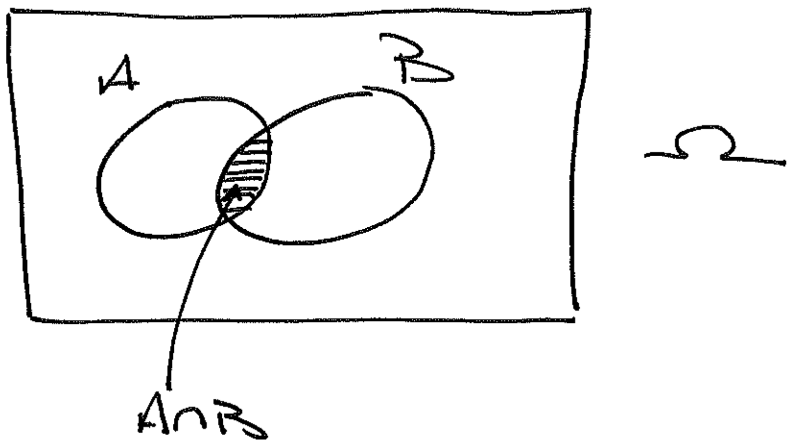
(a) A, B are independent.

(b) A, B^c " " .

(c) A^c, B " " .

(d) A^c, B^c " " .

Geometric intuition



If Probability is normalized area in Ω, then A, B independent means

$$P(A|B) = P(A)$$

$$\frac{P(A\cap B)}{P(B)} = P(A) = \frac{P(A)}{P(\Omega)}$$

$$\frac{\text{area}(A\cap B)}{\text{area}(B)} = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

Conditional Independence

Recall if $P(C) > 0$, then $P(\cdot \mid C)$ is a valid Prob. law.

Defn
We say A, B are conditionally independent given C iff

$$\boxed{P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)}$$

interesting equivalent definition:

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)}$$

mult. law

$$= \frac{\cancel{P(C)} \cdot P(B \mid C) \cdot P(A \mid B \cap C)}{\cancel{P(C)}}$$

$$= P(B \mid C) \cdot P(A \mid B \cap C)$$

A, B cond. ind. is equiv. to

$$P(A|C)\cdot \cancel{P(B|C)} = \cancel{P(B|C)}\cdot P(A|B\cap C)$$

which is equiv. to

$$\boxed{P(A|B\cap C) = P(A|C)}$$

<u>note</u>:

what does $P(A|B|C)$ mean?

$$P(A|B|C) = \frac{P(A\cap B|C)}{P(B|C)}$$

$$= \frac{\dfrac{P(A\cap B\cap C)}{\cancel{P(C)}}}{\dfrac{P(B\cap C)}{\cancel{P(C)}}}$$

$$= \frac{P(A\cap B\cap C)}{P(B\cap C)} = P(A|B\cap C)$$

warning:

cond. indep of A, B w.r.t. C	$\not\Rightarrow$ $\not\Leftarrow$	unconditional independence of A, B

see examples 1.20 and 1.21 in text p. 37

<u>Indep. of multiple events</u>

say we have 3 events: $A, B, C \subseteq \Omega$

<u>Defn</u>
we say A, B, C are independent iff

$$(1)\ P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

Pairwise indep.
$$\begin{cases} (2)\ P(A \cap B) = P(A) \cdot P(B) \\ (3)\ P(B \cap C) = P(B) \cdot P(C) \\ (4)\ P(A \cap C) = P(A) \cdot P(C) \end{cases}$$

Warning

$$\boxed{\begin{array}{l}\text{Pairwise independence}\\ \text{of } A, B, C\\ \text{i.e. } (2),(3),(4)\end{array}} \;\begin{array}{c}\not\Longrightarrow\\[1em] \not\Longleftarrow\end{array}\; \boxed{\begin{array}{l}(1)\ P(A\cap B\cap C)\\ \quad = P(A)\cdot P(B)\cdot P(C)\end{array}}$$

see examples 1.22, 1.23 in text p. 39.

<u>Defn</u>

We say $A_1, A_2, \dots, A_n$ are <u>independent</u> iff

$$P(A_{i_1}\cap A_{i_2}\cap \dots \cap A_{i_k}) = P(A_{i_1})\cdot P(A_{i_2})\cdots P(A_{i_k})$$

for every $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, 3, \dots, n\}$

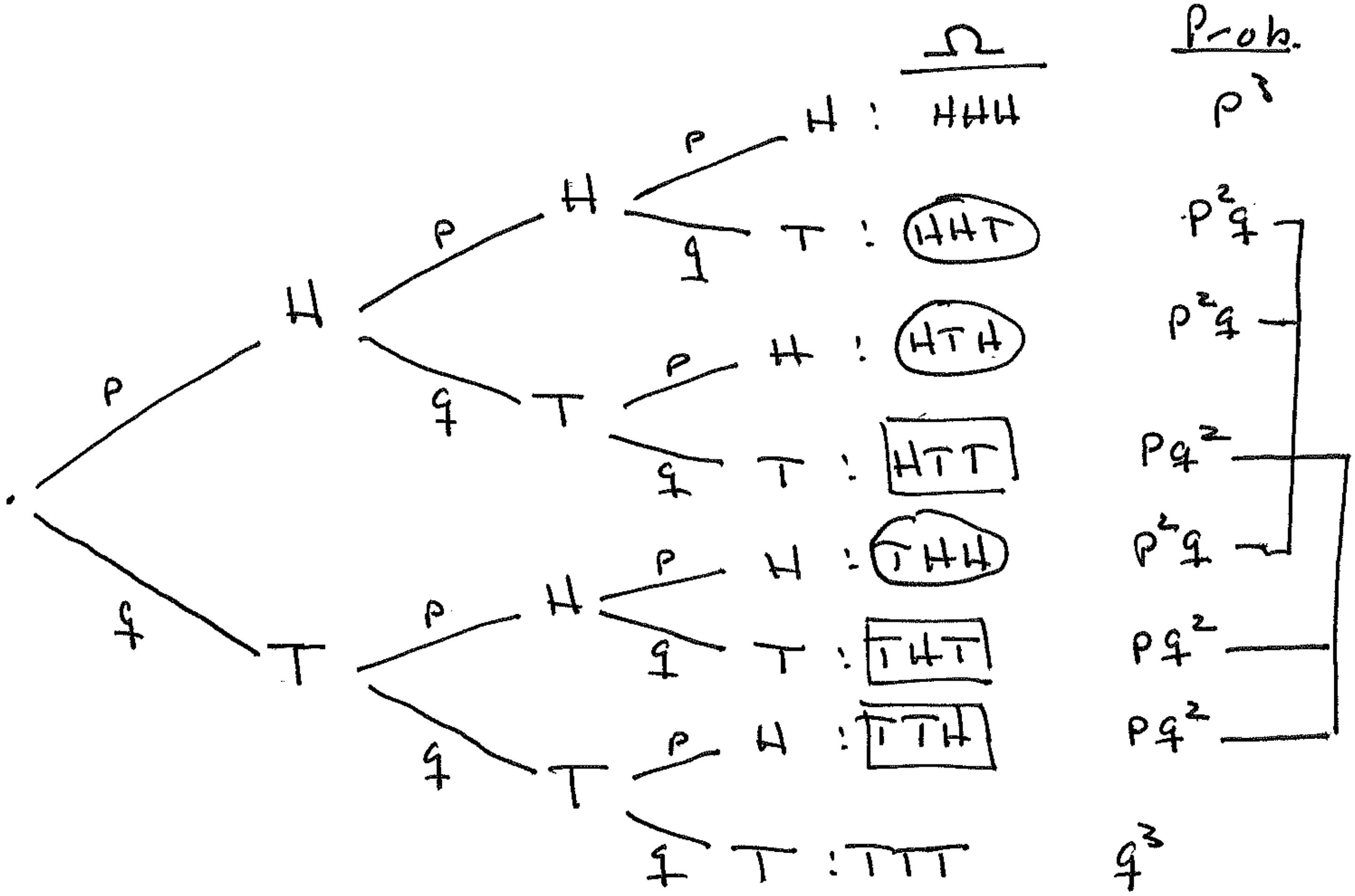
Binomial Probabilities

Can compose several independent experiments to get a new experiment.
subexperiments are called trials.

A Bernoulli Trial is a simple experiment with exactly 2 outcomes

$$\Omega = \{\text{success}, \text{failure}\}$$
$$= \{\text{True}, \text{False}\}$$
$$= \{\text{Heads}, \text{Tails}\}$$
$$= \{1, 0\}$$

Ex. 3 independent tosses of a weighted coin: $P(H) = p$, $P(T) = q = 1-p$



note

$$p^3 + 3p^2q + 3pq^2 + q^3 = (p+q)^3 = 1^3 = 1$$

$$\binom{3}{0}p^3 + \binom{3}{1}p^2q + \binom{3}{2}pq^2 + \binom{3}{3}q^3$$

$$P(\#\text{heads} = 2) = 3 \cdot p^2 q$$

$$P(\#\text{heads} = 1) = 3 \cdot p q^2$$

In general, n flips of a weighted coin, with $P(\text{head}) = p$, then

$$\boxed{P(\#\text{heads} = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Binomial Probabilities

$(1 \le k \le n)$

Note: $\binom{n}{k}$ is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where

$$\binom{n}{k} = \boxed{\begin{array}{l}\#\text{ of } k\text{-subsets}\\ \text{of an } n\text{-set}\end{array}} = \boxed{\begin{array}{l}\#\text{ of bit strings}\\ \text{of len. } n \text{ having}\\ \text{exactly } k \text{ 1's}\end{array}}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

1.6 Combinatorics

read 1.6 : cse 16 review

- # permutations of n objects : $n!$
- # k-permutations of n objects : $\frac{n!}{(n-k)!}$
- # k-combinations of n objects (no repetition)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \boxed{\text{\# k-subsets of an n-set}} = \boxed{\text{\# bit strings of len. n with k 1's}}$$

- # k-combinations of n objects (with repetition)

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

Ex. $n=4$, $k=2$, $S=\{1,2,3,4\}$

Bit strs. len 4	subsets of S
0000	ϕ
0001	$\{4\}$
0010	$\{3\}$
0011	$\{3,4\}$
0100	$\{2\}$
0101	$\{2,4\}$
0110	$\{2,3\}$
0111	$\{2,3,4\}$
1000	$\{1\}$
1001	$\{1,4\}$
1010	$\{1,3\}$
1011	$\{1,3,4\}$
1100	$\{1,2\}$
1101	$\{1,2,4\}$
1110	$\{1,2,3\}$
1111	$\{1,2,3,4\}$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!\cdot 2!} = \frac{\cancel{4}^{2}\cdot 3\cdot \cancel{2}}{\cancel{2}\cdot\cancel{2}} = 6 .$$