CSE 107 1-16-24

✓ • hw1 extended

✓ • disease example

✓ • lab1 posted

Recall

• $P(A \cap B) = P(A) \cdot P(B|A)$

• $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

Proof

$$RHS = \cancel{P(A)} \cdot \frac{\cancel{P(A \cap B)}}{\cancel{P(A)}} \cdot \frac{P(A \cap B \cap C)}{\cancel{P(A \cap B)}}$$

$$= P(A \cap B \cap C) = LHS. \quad \blacksquare$$

• exercise:

$$P(A \cap B \cap C \cap D) = P(A) \cdot P(B|A) \cdot P(C|A \cap B) \cdot P(D|A \cap B \cap C)$$

Multiplication rule

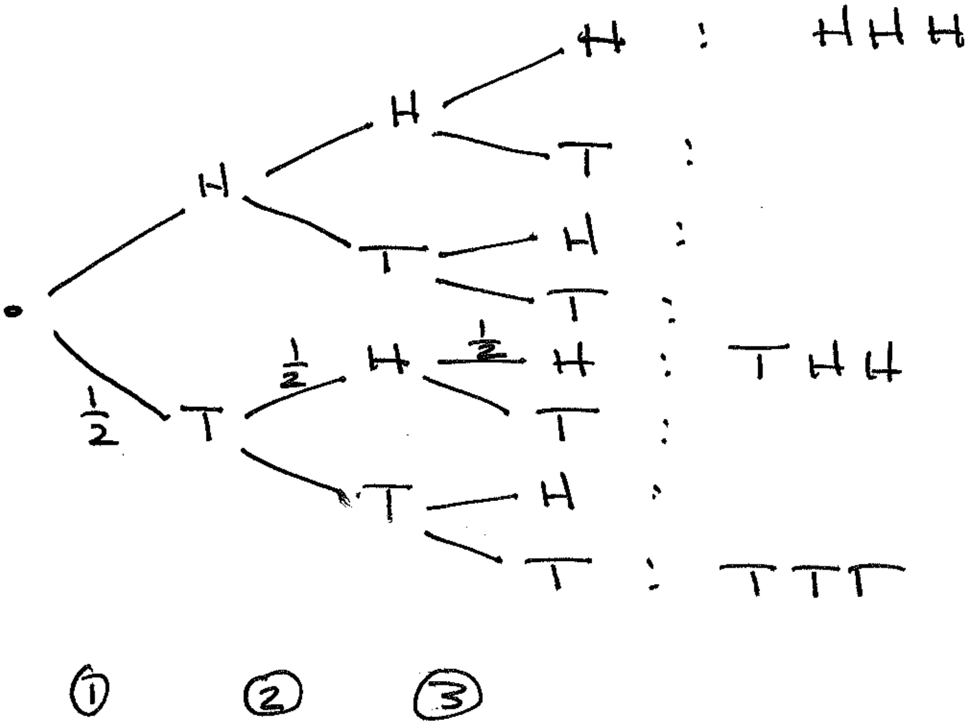
Let $A_1, A_2, \ldots, A_n \subseteq \Omega$. Then

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2)\cdots$$

$$\cdots P\left(A_n \,\middle|\, \bigcap_{i=1}^{n-1} A_i\right)$$

This rule justifies multiplying probabilities of branches along a tree diagram

Ex toss 3 fair coins — Probability



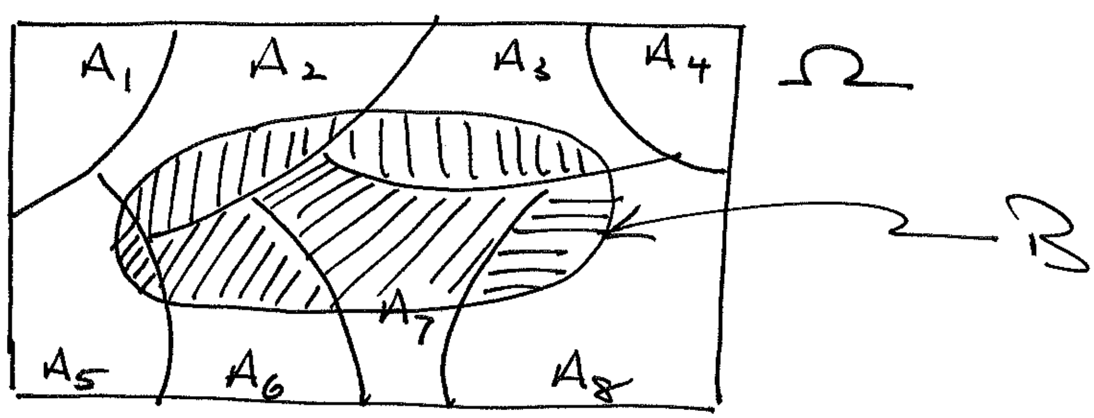
$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{8}}$

1.4 Total Probability theorem, Bayes rule

Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint events whose union is Ω

$$\Omega = A_1 \cup A_2 \cup \cdots \cup A_n$$

i.e. $A_1, \ldots, A_n$ form a Partition of Ω.



Let $B \subseteq \Omega$. observe

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$

Total Probability theorem

$$\boxed{P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \dots + P(A_n \cap B)}$$

If $P(A_i) > 0$ for all $i = 1, \dots, n$, then

$$\boxed{P(B) = P(A_1) \cdot P(B|A_1) + \dots + P(A_n) \cdot P(B|A_n)}$$

<u>special case</u>: Partition is A, A^c

$$\boxed{P(B) = P(A) \cdot P(B|A) + P(A^c) \cdot P(B|A^c)}$$

<u>Ex.</u> Roll a fair 6-sided die. If result is in $\{1, 2, 3\}$ roll again, otherwise stop. What is the prob. that the sum of all rolls is ≥ 5.

Let $B = \{\text{sum} \geq 5\}$. Let

$$A_i = \{1^{st} \text{ roll is } i\} \quad (1 \leq i \leq 6)$$

<u>note</u>: A_i form a partition of Ω.

<u>also</u>:

if A_1 occurs, then B occurs iff 2^{nd} roll $\in \{4,5,6\}$
" A_2 " " " " " " " $\in \{3,4,5,6\}$
" A_3 " " " " " " " $\in \{2,3,4,5,6\}$

Thus

$$P(B|A_1) = \frac{3}{6} = \frac{1}{2}$$

$$P(B|A_2) = \frac{4}{6} = \frac{2}{3}$$

$$P(B|A_3) = \frac{5}{6}$$

$$P(B|A_4) = 0$$

$$P(B|A_5) = 1$$

$$P(B|A_6) = 1$$

Hence:

$$P(B) = P(A_1) \cdot P(B|A_1) + \dots + P(A_6) \cdot P(B|A_6)$$

$$= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{2}{3} + \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1$$

$$= \dots = \boxed{\frac{2}{3}}$$

Bayes Rule

Let $A_1, A_2, \ldots, A_n$ form a Partition of Ω. Assume $P(A_i) > 0$ for $i = 1, \ldots, n$. Assume $P(B) > 0$. Then for $i = 1, \ldots, n$

$$\boxed{P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(B)}} \quad (1)$$

and

$$\boxed{P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(A_1) \cdot P(B \mid A_1) + \cdots + P(A_n) \cdot P(B \mid A_n)}} \quad (2)$$

<u>note</u>: (2) follows from (1) using total Prob. thm.

To prove (1) recall

$$P(B)\cdot P(A|B) = P(A\cap B) = P(A)\cdot P(B|A)$$

hence

$$P(A|B) = \frac{P(A)\cdot P(B|A)}{P(B)} \quad \blacksquare$$

How do we use Bayes rule?

suppose we have causes

$$A_1, A_2, \ldots\ldots, A_n$$

that may result in an effect B.

we observe the effect, and try to infer the causes

$$\begin{array}{ccc} A_1 & \searrow & \\ A_2 & \longrightarrow & B \\ \vdots & \vdots & \\ A_n & \nearrow & \\ \text{causes} & & \text{effect.} \end{array}$$

$P(B|A_i)$ is a model of the strength of the causal relation

$$A_i \xrightarrow[\text{causation}]{} B \quad : P(B|A_i)$$

Given B is observed, we wish to evaluate $P(A_i|B)$.

i.e. go from effect to cause

$$A_i \xleftarrow[\text{inference}]{\qquad} B \;:\; P(A_i|B)$$

$$\underbrace{P(A_i|B)}_{\text{Posterior}} = \frac{\overbrace{P(A_i)}^{\text{Prior}} \cdot \overbrace{P(B|A_i)}^{\text{likelyhood}}}{\underbrace{P(B)}_{\text{marginal}}}$$

Ex. Disease example again:

$T = \{\text{test positive}\}$

$D = \{\text{has disease}\}$

$P(T|D) = .99$

$P(T|D^c) = .10$

$P(D) = .05 \qquad P(D^c) = .95$

Apply Bayes rule with $A_1 = D$, $A_2 = D^c$ to find

$$P(D|T) = \frac{P(D)\cdot P(T|D)}{P(T)}$$

$$= \frac{P(D)\cdot P(T|D)}{P(D)\cdot P(T|D) + P(D^c)\cdot P(T|D^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99)+(.95)(.10)} = \boxed{.3426}$$

round to 4 digits

1.5 Independence

It's possible that knowledge that B occurs tells us <u>nothing</u> about whether A occurs:

$$(*) \qquad P(A|B) = P(A)$$

in this case, we say A is <u>independent</u> of B. since

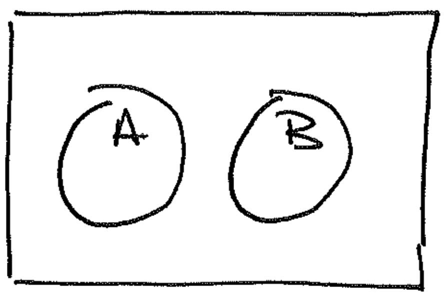
$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

(*) is equivalent to

$$(**) \qquad P(A\cap B) = P(A)\cdot P(B)$$

<u>Defn</u> we say $A, B \subseteq \Omega$ are <u>independent</u> iff $\boxed{P(A\cap B) = P(A)\cdot P(B)}$

note: $A \cap B = \phi \not\Longleftrightarrow A, B$ independent



$A \cap B = \phi$

Ex. Throw 2 fair dice. let

$A_i = \{ 1^{st} \text{ die is } i \} \quad 1 \le i \le 6$

$B_j = \{ 2^{nd} \text{ die is } j \} \quad 1 \le j \le 6$

If A_i, B_j are independent, then

$P((i,j)) = P(A_i \cap B_j) = P(A_i) \cdot P(B_j) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

we get the Discrete Uniform Prob. law.