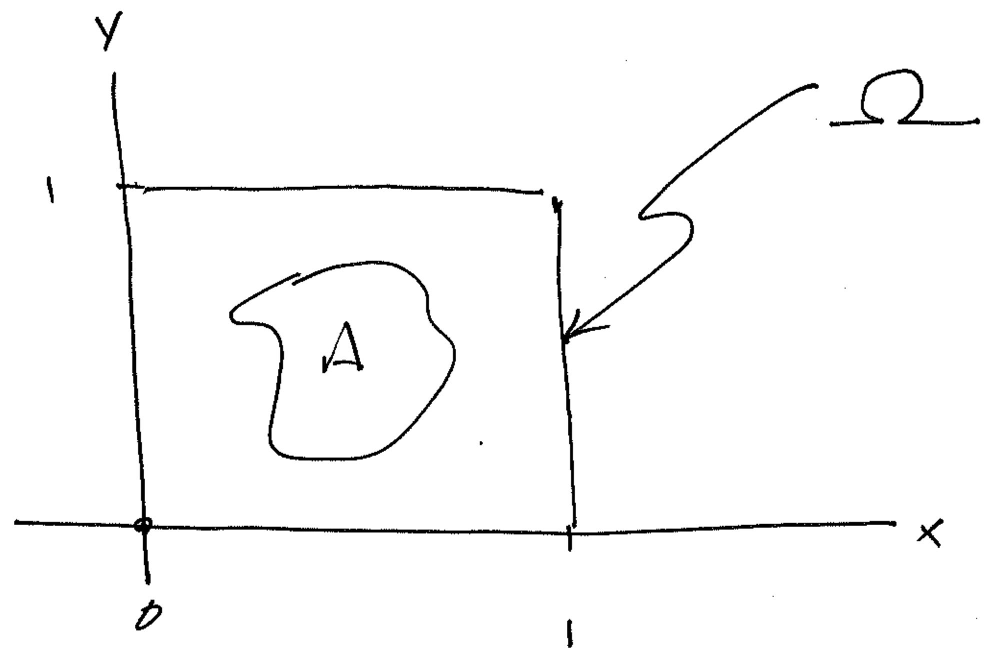
CSE 107 1-11-24

Continuous models

Ω is $[a,b]$, region in $\mathbb{R}^2$, or in $\mathbb{R}^3$

Ex. R. & J have a date, each will be delayed by a random amount in $[0,1]$, units = hours. let x be R.'s delay time and y J.'s delay. The 1^{st} to arrive will wait 15 min. then leave, if the other has not arrived. What is the prob. that they meet?

We assume all delay pairs (x, y) are 'equally likely'.



This means that

$$P(A) = \text{const} \cdot \text{area}(A)$$

for any region $A \subseteq \Omega$.

<u>note</u>: $P(\Omega) = 1 \quad \therefore \quad 1 = \text{const} \cdot \text{area}(\Omega)$

$\therefore \; 1 = \text{const} \cdot 1 \quad \therefore \; \boxed{\text{const} = 1}$

<u>note</u>: $P((x,y)) = 0$

We seek $P(M)$ where

$M = \{R \text{ \& } J \text{ meet}\}$

$$M = \{(x,y) \mid |x-y| \le \tfrac{1}{4},\ (x,y) \in \Omega\}$$

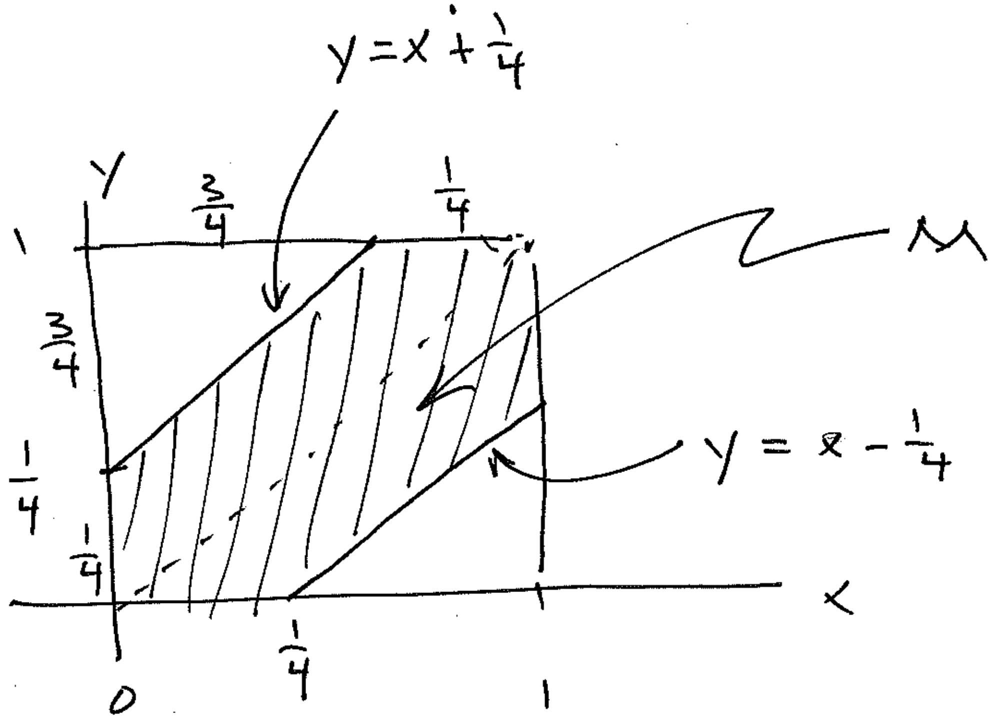
observe $(x,y) \in M$ iff

$$|x-y| \le \frac{1}{4}$$

$$-\frac{1}{4} \overset{①}{\le} x-y \overset{②}{\le} \frac{1}{4}$$

$\therefore$ ① $y \le x + \frac{1}{4}$ and $y \ge x - \frac{1}{4}$

so



$$P(M) = \text{area}(M) = 1 - \left(\frac{3}{4}\right)^2$$

$$= 1 - \frac{9}{16}$$

$$= \boxed{\frac{7}{16}}$$

Exercise

Alice & Bob each pick a rand. number in range $[0, 2]$.

Alice : x
Bob : y

what is the prob. that both

$$2x > y^2 \text{ and } 2y > x^2$$

Properties of $P(\cdot)$

One can prove

(a) if $A \subseteq B$, then $P(A) \leq P(B)$

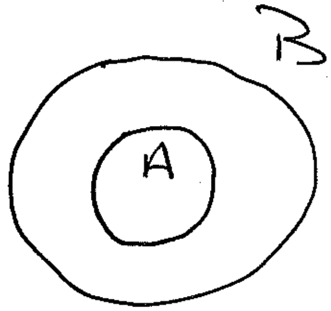
(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(c) $P(A \cup B) \leq P(A) + P(B)$

(d) $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

Proof of (a)

$$A \subseteq B \Rightarrow \begin{cases} B = A \cup (B-A) \\ A \cap (B-A) = \phi \end{cases}$$

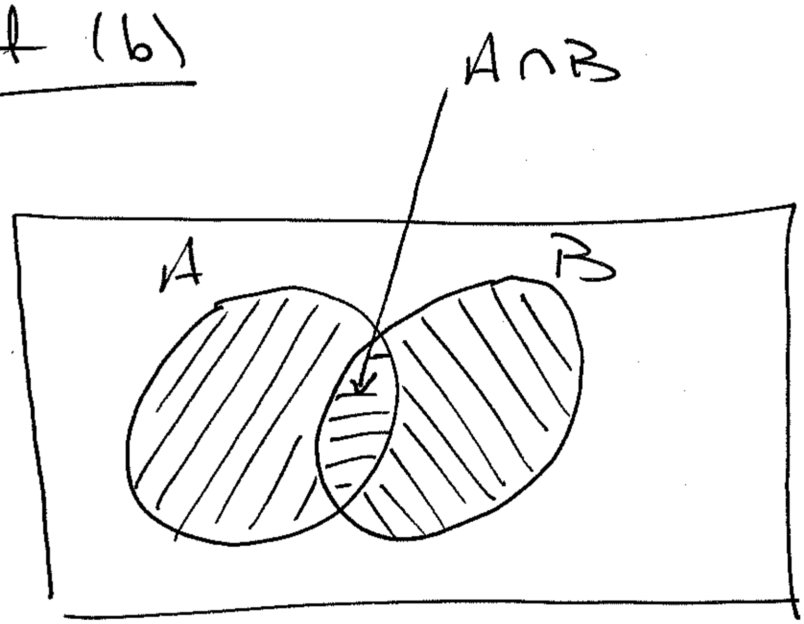


Thus

$$P(B) = P(A) + P(B-A) \geq P(A)$$

$\therefore$ $P(A) \leq P(B)$. $\blacksquare$

Proof of (b)



note: $A \cup B = \underbrace{(A-B) \cup (A \cap B) \cup (B-A)}_{\text{Pairwise disjoint}}$

$\therefore P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$

by axiom 3.

note: $P(B) = P(B-A) + P(A \cap B)$

$\therefore \; P(B-A) = P(B) - P(A \cap B)$.

similarly:

$$P(A-B) = P(A) - P(A \cap B).$$

Thus

$$P(A \cup B) = \left(P(A) - \cancel{P(A \cap B)}\right) + \cancel{P(A \cap B)} + \left(P(B) - P(A \cap B)\right)$$

$$= P(A) + P(B) - P(A \cap B)$$

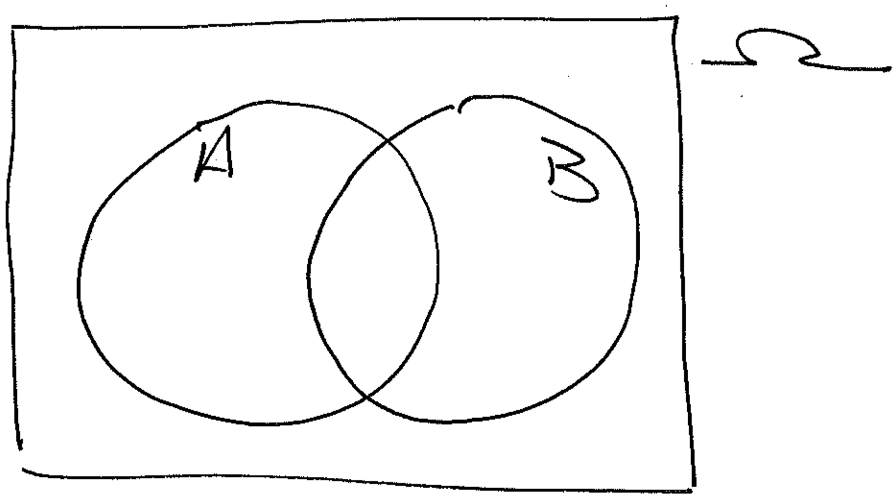
$\blacksquare$

note: (c) follows from (b)

Proof of (d): exercise.

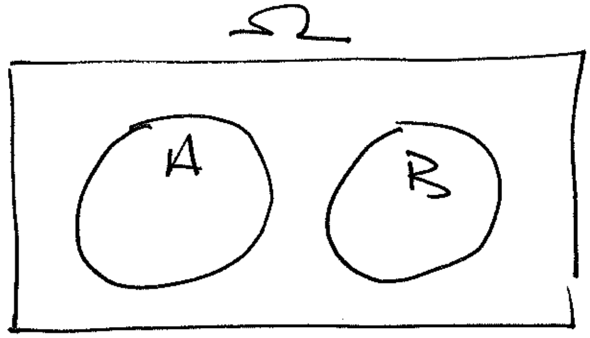
1.3 Conditional Probability

Suppose A, B are events in Ω



How does $P(A)$ change if we know B occurs.

Extreme cases

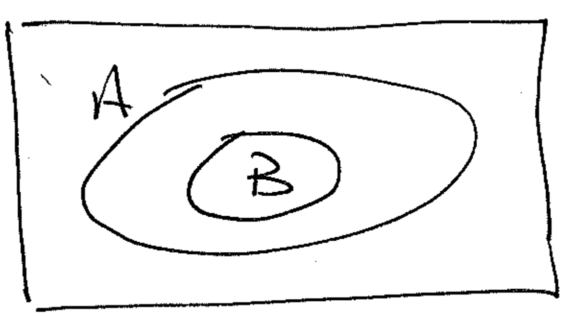


$A \cap B = \phi$

$\therefore \; A \subseteq B^c$

$B \subseteq A^c$

B occurs $\Rightarrow$ A does not occur



$B \subseteq A$

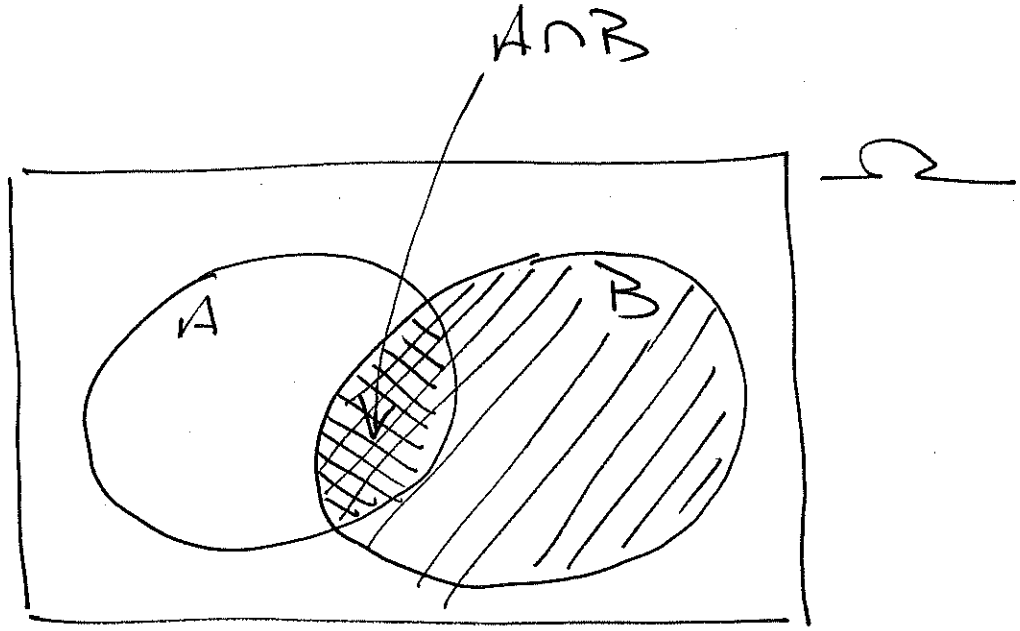
B occurs $\Rightarrow$ A does occur

Defn

The <u>conditional Probability</u> of A given B, denoted $P(A|B)$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

(where we assume $P(B) > 0$.)



observe that $P(\cdot \mid B)$ is itself a valid probability law.

<u>check Axioms</u>

(i) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \geq 0$

(ii) $P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(iii) if $A_1 \cap A_2 = \phi$, then

$$P(A_1 \cup A_2 | B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 | B) + P(A_2 | B). \quad \blacksquare$$

Thus any theorm that holds for $P(\cdot)$ also holds for $P(\cdot | B)$

For instance

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

becomes

- $P(A_1 \cup A_2 | B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$

Also: if Ω is finite, $P(\cdot)$ is uniform, then

- $P(A|B) = \dfrac{|A \cap B|}{|B|}$ (exercise)

The defn: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Gives us

$$\boxed{P(A \cap B) = P(A|B) \cdot P(B)}$$

Ex.

A test for a certain disease is Positive with Prob. .99, if the subject has the disease, and Positive with Prob. .10, if the subject does not have disease. The Prob. that a random individual has the disease is .05.

- What is Prob. false Positive?
- What " " false negative?

let

$T = \{\text{test is positive}\}$

$D = \{\text{subject has disease}\}$

we are given:

$P(T|D) = .99 \quad \therefore P(T^c|D) = .01$

$P(T|D^c) = .10 \quad \therefore P(T^c|D^c) = .90$

$P(D) = .05 \quad \therefore P(D^c) = .95$

so

$P(T \cap D^c) = P(T|D^c) \cdot P(D^c)$

$= (.10) \cdot (.95) = \boxed{.095}$

and

$$P(T^c \cap D) = P(T^c | D) \cdot P(D)$$

$$= (.01)(.05) = \boxed{.0005}$$

<u>Observe</u>

- $P(A|B) \cdot P(B) = \underline{P(A \cap B) = P(B|A) \cdot P(A)}$

also

- $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$