CSE 107
Homework Assignment 7

Section 4.1: Derived Distributions

1. Let X be exponentially distributed with parameter $\lambda = 1$, and let Y be uniformly distributed over the interval $[0, 1]$. Use convolution to find the PDF of $Z = X - Y$.

2. Let X be a discrete random variable with PMF $p_X(x)$, and let Y be a continuous random variable, independent from X, with PDF $f_Y(y)$. Derive a formula for the PDF of the continuous random variable $Z = X + Y$.

3. Recall if X and Y are random variables, then the notation $X \sim Y$ means $F_X = F_Y$. Let X and Y be **independent**, and continuous uniform on the interval $[0,1]$. Suppose $Z = \min(X, Y)$ and let $W = 1 - \sqrt{1 - X}$. Prove that $Z \sim W$.

Section 4.2: Covariance and Correlation

4. Let X, Y and Z be random variables whose means are 0, variances are 1, and which are pairwise uncorrelated (i.e. $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, Z) = \mathrm{Cov}(Y, Z) = 0$). Find $\rho(U, V)$ where $U = X + Y$ and $V = Y + Z$.

Section 4.3: Conditional Expectation and Variance

5. Let n be a positive integer, R_n be a continuous uniform random variable on the interval $[1/n,\ 1]$, and X_n be a geometric random variable whose parameter is itself the random variable R_n. Find $E[X_n]$ and $\lim_{n\to\infty} E[X_n]$. (Hint: use the law of iterated expectations: $E[X_n] = E\big[E[X_n|R_n]\big]$.)

Section 4.5: The Sum of a Random Number of Random Variables

6. Given two fair 6-sided dice and a fair coin:

   a. Roll one die, then flip the coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained.

   b. Roll both dice, observe the sum of the faces showing, then flip the coin the number of times given by the sum. Find the expected value and variance of the number of heads obtained.

Section 6.1: The Bernoulli Process

7. Consider a Bernoulli process with parameter p.

   a. Let T be the (discrete) time until the first arrival. Write down the PMF of T.

   b. Let N_t be the number of arrivals in the time interval $[1, t]$ (where t is a positive integer). Write down the PMF of N_t.

   c. Let Y_k be the time until the k^{th} arrival. Write down the PMF of Y_k.

   d. Let H be the number of arrivals that occur in the time interval $[1, 20]$, and let C be the event that exactly 10 arrivals occur in the time interval $[1, 18]$. Determine the conditional expectation $E[H|C]$ and the conditional variance $\text{Var}(H|C)$. (Hint: each number depends on p.)

   e. Now suppose there is a second, independent Bernoulli process, also with parameter p, running parallel to the first (i.e. running on the same discrete time clock.) Consider the merged process in which an arrival occurs whenever there is an arrival in the first process, or in the second, or in both. Determine the probability that the 5^{th} arrival occurs at time 10.

8. Let Y_{17} be the Pascal random variable of order 17 and parameter p. Determine positive integers a and b such that

$$\sum_{j=42}^{\infty} p_{Y_{17}}(j) = \sum_{k=0}^{a} \binom{b}{k} p^k (1-p)^{b-k}$$

   (Hint: consider a Bernoulli process of parameter p, and identify both sides of the above equation as the probability of one and the same event.)

9. Consider a Bernoulli process of parameter p, and let Y_k be the (discrete) time of the k^{th} arrival, which is a Pascal random variable of order k. Let X_k be the number of ticks of the clock, before the k^{th} arrival, in which no arrival occurred. Using a different analogy, Y_k is the number of trials until the k^{th} success, and X_k is the number of failures before the k^{th} success. Better yet, Y_k is the number of coin flips until the k^{th} head, and X_k is the number of tails before the k^{th} head. (X_k is called the **Negative Binomial** random variable of order k.)

   a. Determine the PMF $p_{X_k}(t)$ of X_k. (Hint: observe that $X_k = Y_k - k$.)

   b. Determine the mean $E[X_k]$.

   c. Determine the variance $\text{Var}(X_k)$.

Section 6.2: The Poisson Process

10. A single light bulb illuminates a room starting at time $t = 0$, and is replaced immediately upon failure. The process of replacing bulbs when they burn out continues indefinitely. Each new bulb is selected independently, with equal probability, to be of type-A or of type-B. The initial bulb is also equally likely to be of type-A or of type-B. The lifetime T of any particular bulb is a random variable with one of two possible (exponential) PDFs.

$$\text{type-A:} \qquad f_T(t) = \begin{cases} 2e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

or

$$\text{type-B:} \qquad f_T(t) = \begin{cases} 3e^{-3t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the expected time until the first failure.

b. Find the probability that there are no bulb failures before time t.

c. Given that there are no failures before time t, determine the conditional probability that the first bulb used is of type-A.

d. Find the variance of the time until the first bulb failure.