CSE 107
Homework Assignment 6

Section 3.5: Conditioning

1. Consider a random variable X with PDF

$$f_X(x) = \begin{cases} \frac{2x}{3} & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   and let A be the event $A = \{X \geq 1.5\}$.

   a. Find $E[X]$ and $P(A)$.
   b. Find $f_{X|A}(x)$ and $E[X|A]$.
   c. Find $E[X^2]$ and $\text{Var}(X)$.
   d. Find $E[X^2|A]$ and $\text{Var}(X|A)$.

2. Alice is vacationing in Monte Carlo. The amount of money X she takes to the casino each evening is a random variable with PDF of the form

$$f_X(x) = \begin{cases} ax & \text{if } 0 \leq x \leq 40 \\ 0 & \text{otherwise} \end{cases}$$

   At the end of each night of gambling, the amount Y that she takes back to her hotel room is uniformly distributed between 0 and twice the amount she came with.

   a. Find the value of a in the above formula for $f_X(x)$.
   b. Write down the conditional PDF $f_{Y|X}(y|x)$ based on the given information.
   c. Determine the joint PDF $f_{X,Y}(x,y)$.
   d. Determine the marginal PDF $f_Y(y)$.
   e. Find the expected value of her profit $Z = Y - X$.
   f. What is the probability that on a given night, Alice makes a positive profit at the casino?

3. Pick a random point X in the unit interval $[0,1]$ using a uniform distribution. Pick another random point Y in the interval $[X,1]$, again using a uniform distribution. What is the probability that the three line segments $[0,X]$, $[X,Y]$ and $[Y,1]$ can form the legs of a triangle? (Hint: the three segments form a triangle if and only if each segment is shorter than the sum of the other two, i.e. if and only if $X < (Y - X) + (1 - Y)$, $Y - X < X + (1 - Y)$ and $1 - Y < X + (Y - X)$. Simplify these inequalities and draw the corresponding region in the x-y plane.)

Section 3.6: The Continuous Bayes' Rule

4. Let X have a uniform distribution in the unit interval $[0,1]$, and let Y have an exponential distribution with parameter $\lambda = 2$. Assume that X and Y are independent, and let $Z = X + Y$.

   a. Find $P(Y \geq X)$.
   b. Find the conditional PDF of Z given that $Y = y$.
   c. Find the conditional PDF of Y given that $Z = 3$.

Section 4.1: Derived Distributions

5. Let X be a random variable with PDF $f_X(x)$. Find the PDF of the random variable $Y = e^X$ in terms of $f_X(x)$.

6. Let X be a random variable with PDF $f_X(x)$.

   a. Find the PDF of the random variable $Y = |X|$ in terms of $f_X(x)$.
   b. Find $f_Y(y)$ if $Y = |X|$, and X is exponential with parameter λ.
   c. Find $f_Y(y)$ if $Y = |X|$, and X is continuous uniform on the interval $[-2, 3]$.

7. Let X and Y be independent discrete random variables with the same PMF:

$$p_X(x) = \begin{cases} 1/4 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \\ 1/2 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

   and similarly for Y. Use the convolution product to find the PMF of $Z = X + Y$.

8. Let X be continuous uniform on the interval $[0, 2]$, and Y continuous uniform on $[3, 4]$. Assume that X and Y are independent. Use the convolution product to find the PDF of $Z = X + Y$.

9. Let X and Y be independent random variables, both continuous uniform on the interval $[0, 1]$.

   a. Let $U = \max(X, Y)$, and find the mean $E[U]$ of U.
   b. Let $V = \min(X, Y)$, and find the mean $E[V]$ of V.
   c. Find the mean of the distance between X and Y, i.e. find $E[\,|X - Y|\,]$. (Hint: note that $|X - Y| = U - V$.)