CSE 107
Homework Assignment 2

Section 1.5: Independence

1. You enter a special kind of chess tournament, in which you play one game against each of three opponents, but you get to choose the order in which you play your opponents. You win the tournament if you win two games in a row. You know your probability of a win against each of the three opponents, and that the opponents are of differing strengths. Label the players 1, 2 and 3 by increasing strength, so that player 1 is the weakest and player 3 the strongest. Let p_i be your probability of winning against player i, so that $p_1 > p_2 > p_3$. Assuming that individual matches are independent events, determine the optimal order(s) of play: 123, 132, 213, 231, 312, or 321. Determine your probability of winning the tournament, given the optimal order of play.

2. A parking lot consists of a single row containing n parking spaces ($n \geq 2$). Mary arrives when all spaces are free. Tom is the next person to arrive. Each person makes an equally likely choice among all available spaces at the time of arrival, independent of the other person's choice. Describe the sample space and determine the probability that the parking spaces selected by Mary and Tom are at most 2 spaces apart. (In other words, either Tom's space is next to Mary's, or there is a single space between them.)

3. A particular jury consists of 7 jurors. Each juror has a 0.2 chance of making the wrong decision, independently of the others. If the jury reaches a decision by majority rule, what is the probability that it will reach a wrong decision?

4. Suppose that events A, B, and C are independent.

   (a) Show that A and $B \cup C$ are independent.
   (b) Show that A^c and B^c are independent.
   (c) Show that A^c, B^c and C^c are independent.

5. A 7-sided die is thrown, with faces labeled $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$, and each outcome equally likely. Define events $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$ and $C = \{1, 2, 4, 5, 6, 7\}$.

   (a) Determine whether A and B are independent.
   (b) Determine whether A and B are conditionally independent, given C.
   (c) Determine whether A and C are conditionally independent, given B.

6. A weighted coin has $P(H) = p$ and $P(T) = 1 - p$, where $0 < p < 1$. Suppose the coin is flipped indefinitely (i.e. infinitely many times), and assume all flips are independent. Define a_n to be the probability that after n flips, there have been an even number of heads.

   (a) Determine a_1, a_2 and a_3.
   (b) Determine a recurrence formula for a_n in terms of a_{n-1}, and p (i.e. find a relation of the form $a_n = f(a_{n-1}, p)$ where f is a function that you find explicitly.) Hint: Let A_n be the event that after n flips, there have been an even number of heads, then write an expression for $P(A_n)$ in terms of $P(A_{n-1})$ and $P(A_{n-1}^c)$.
   (c) Use the recurrence you found in part (b) to prove that

$$a_n = \frac{1}{2}(1 + (1 - 2p)^n)$$

Section 2.2: Probability mass functions

7. The UCSC football team will play two games this weekend, each of which can end in a Win, Loss or Tie. It has probability 0.4 of not losing the first game, and probability 0.7 of not losing the second game, independently of the first. When the team does not lose, it is equally likely to win or tie, again independently of the other game. The team earns 2 league credits for each win, 1 credit for each tie, and 0 credits for a loss. Define the random variable X be the number of league credits earned this weekend.

   (a) Determine $\text{range}(X)$.
   (b) Determine the PMF $p_X(k)$ of the random variable X.

Section 2.3: Functions of a random variable

8. Let X be a discrete random variable that is uniformly distributed over the set of integers in the range $[a, b]$, where a and b are integers with $a < 0 < b$. Find the PMF of the random variables $\max(0, X)$ and $\min(0, X)$.

9. Let X be a discrete random variable, and let $Y = |X|$.

   (a) Assume that the PMF of X is

$$p_X(x) = \begin{cases} cx^2 & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

   where c is a suitable constant. Determine the value of c.

   (b) For the PMF of X given in part (a) calculate the PMF of Y.

   (c) Give a general formula for the PMF of $Y = |X|$ in terms of the PMF of X.