# CSE 107
# Midterm 2 Review Problems

1. Alice's drive time from work to home is such that in *light traffic*, she takes between 1 hour and 75 minutes, uniformly distributed. In *heavy traffic* her drive time is uniformly distributed between 75 minutes and 95 minutes. Assume that traffic is equally likely to be light or heavy.

   a. Let $X$ be Alice's commute time, in minutes. Determine the PDF $f_X(x)$.
   b. Determine Alice's expected commute time.

2. Bob throws a dart at a circular target of radius $r$. He hits the target with certainty, but is equally likely to hit any point within the target. Let $Z$ be the distance from Bob's dart to the center of the target.

   a. Find the CDF $F_Z(z)$ and the PDF $f_Z(z)$.
   b. Find the mean $E[Z]$.
   c. Find the variance $Var(Z)$.

3. A city's temperature in degrees Celsius is modeled as a normal random variable $X$ with mean 10 and standard deviation 10. Let $Y$ be its temperature in Fahrenheit, where $X$ and $Y$ are related by

$$X = \frac{5(Y-32)}{9}.$$

   What is the probability that the temperature is above 77 degrees Fahrenheit?

4. Let $X$ be a normal random variable with mean $\mu$ and standard deviation $\sigma$.

   a. Find the probability that $X$ is within $1.25\sigma$ of its mean $\mu$, i.e. find

$$P(\mu - 1.25\sigma \leq X \leq \mu + 1.25\sigma)$$

   b. Determine $c \in \mathbb{R}$ for which the probability that $X$ is within $c\sigma$ of its mean $\mu$ is $0.8294$, i.e. find $c$ such that

$$P(\mu - c\sigma \leq X \leq \mu + c\sigma) = 0.8294$$

5. Let $Y$ be a normal random variable with variance 1, and with mean another random variable $X$. Suppose $X$ is continuous uniform on the interval $[1, 3]$.

   a. Find the PDF $f_Y(y)$.
   b. Find the conditional PDF $f_{X|Y}(x|y)$.
   c. Suppose we sample $Y$ and get $Y = 3$. What is the probability that $X \leq 2$ ?
   d. Find $E[Y]$.

6. Let $X$ and $Y$ be jointly continuous random variables, and suppose

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & \text{if } 0 < y \leq 1 \text{ and } 1 - y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Hint: Before you do the following problems, draw a picture of the region defined by the inequalities $0 < y \leq 1$ and $1 - y \leq x \leq 1$.

a. Determine the joint PDF $f_{X,Y}(x,y)$.
b. Determine the marginal PDF $f_X(x)$.
c. Determine the expected value $E[X]$.
d. Determine the conditional expectation $E[X|Y = y]$.

7. Let $X$ be an exponential random variable with parameter $\lambda$, and let $Y = X + 1$. Determine the PDF $f_Y(y)$.

8. Alice is at the casino with a choice of two games. The first returns winnings (positive or negative) that are normally distributed with parameters $\mu = 1$ and $\sigma = 2$. The second is uniformly distributed with winnings in the range $-1$ to $2$. (All amounts are in dollars.) She flips a coin with $P(\text{head}) = p$ to decide which game to play. If heads, she plays the first game, and if tails, she plays the second. Determine her expected winnings, in terms of $p$.

9. Let $X$ and $Y$ be independent, jointly continuous random variables, where $X$ is uniform on $[a, b]$ and $Y$ is uniform on $[c, d]$.

a. Write the PDFs $f_X(x)$ and $f_Y(y)$.
b. Write the CDFs $F_X(x)$ and $F_Y(y)$.
c. Let $Z = \max(X, Y)$. Determine the PDF $f_Z(z)$. (Assume $a \leq d$ and $c \leq b$, so that the two intervals overlap.)
d. Let $Z = X + Y$. Determe the PDF $f_Z(z)$.

10. A point is chosen at random (i.e. uniformly) from within the triangle

$$\{\,(x, y) \mid 0 \leq y \leq x \leq 1\,\}$$

(Hint: draw this triangle before you go any further.)

a. Find the joint PDF $f_{X,Y}(x,y)$
b. Find the marginal PDF $f_Y(y)$
c. Find the conditional PDF $f_{X|Y}(x|y)$
d. Find the conditional expectation $E[X \mid Y = y]$

11. Let $X$ be a uniform continuous random variable on $[1, 5]$, and let $Y = \ln(X)$. Find the PDF $f_Y(y)$. (Hint: use the 2-step process for finding a derived distribution.)

12. Let $X$ and $Y$ be exponential random variables with parameters $\lambda_1 = 2$ and $\lambda_2 = 3$ respectively. Assume $X$ and $Y$ are independent. Use the convolution product to find the PDF $f_Z(z)$ of the random variable $Z = X + Y$.

13. Suppose that $X$ and $Y$ are random variables with $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Prove that $X + Y$ and $X - Y$ are uncorrelated.

14. Let $m \geq 1$ be an integer, let $X$, $Y$ be independent, discrete, uniform random variables on $\{1, 2, \dots, m\}$, and let $Z = X + Y$. Use the convolution product to find the PMF $p_Z(n)$ of $Z$. Hint: observe that $\mathrm{range}(Z) = \{2, 3, \dots, 2m\}$.