

CSE 107 6-4-26

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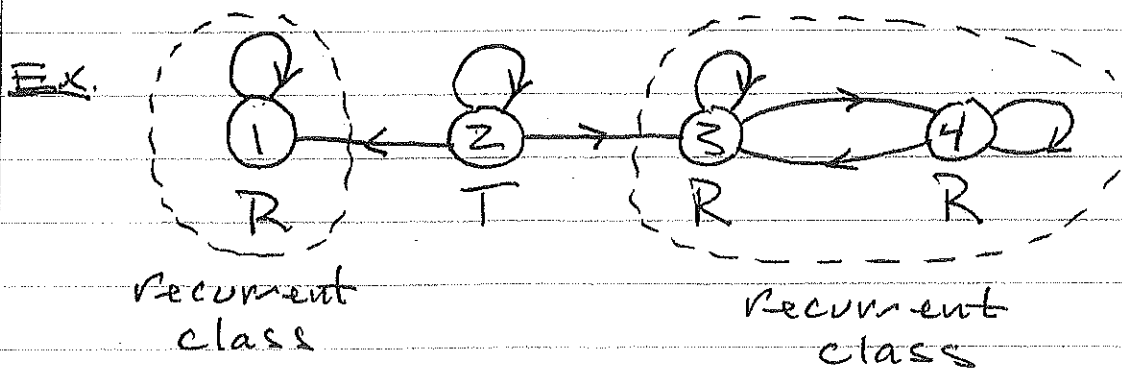
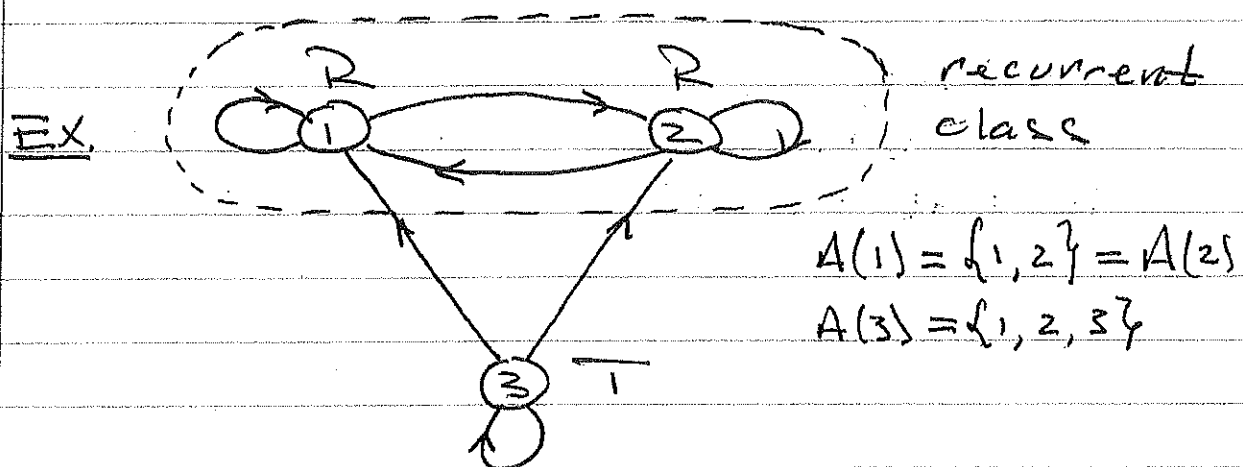
Final : Th. June 11, 4-6 PM

SETS : closes Sun. June 7, 11:59 PM

Thus after a visit to a transient state i , there is a Positive Probability of entering such a state i . Given enough time, this is certain to happen.

Hence a transient state will only be visited a finite number of times.

Note: Transience and recurrence are determined by the arcs in a state transition diagram, not by the actual probabilities P_{ij} on each arc.



Note that for any states i, j we have

$$j \in A(i) \Rightarrow A(i) \subseteq A(j)$$

and

$$i \in A(j) \Rightarrow A(i) \subseteq A(j)$$

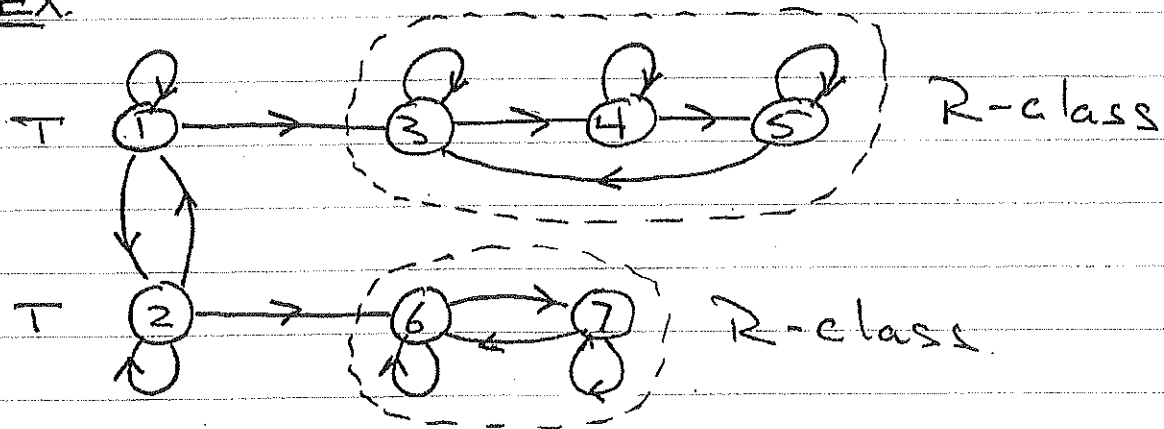
Therefore if i is recurrent, then

$$\left. \begin{array}{l} j \in A(i) \Rightarrow A(i) \subseteq A(j) \\ \Updownarrow \\ i \in A(j) \Rightarrow A(i) \subseteq A(j) \end{array} \right\} \Rightarrow A(i) = A(j)$$

Defn

If i is a recurrent state, the set $A(i) \subseteq S$ is called a recurrent class.

EX.



Problem

Exercise (See #6 on p. 381)

Let i be a transient state. Prove there exists a recurrent state j such that

$$T \text{ (i)} \dashrightarrow \text{(j)} R$$

i.e.

$$j \in A(i)$$

Theorem: Markov Chain Decomposition

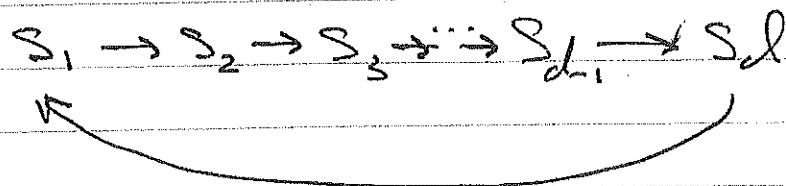
- A Markov chain can be decomposed into one or more recurrent classes, and possibly some transient states.
- A recurrent state is accessible from all states in its class, but not from recurrent states in other classes.
- A transient state is not accessible from any recurrent state.
- At least one recurrent state is accessible from a given transient state.

Theorem

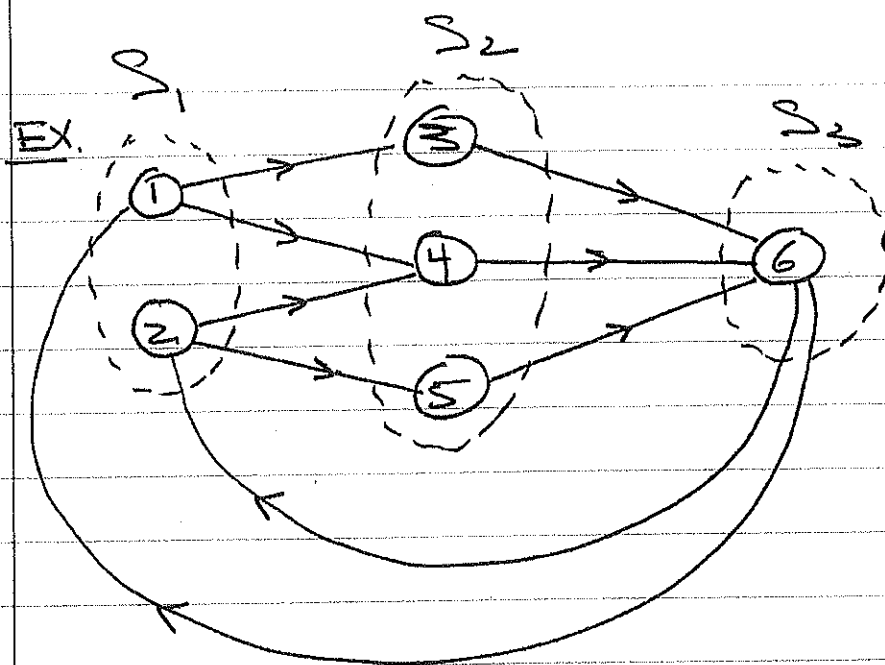
- (a) when a Markov chain enters a recurrent class, it stays within that class, and all states in the class will be visited infinitely often
- (b) If the initial state of a Markov chain is transient, then its trajectory consists of a finite sequence of transient states, followed by an infinite sequence of recurrent states belonging to the same class.

Defn

A recurrent class is called periodic if its states can be grouped into $d \geq 2$ disjoint subsets $S_1, \dots, S_d$ such that all transitions from S_k lead to S_{k+1} ($1 \leq k < d$) and S_d leads to S_1 .



we call d the period of the class



observe that in this example, if $X_0 \in S_1$, then

$$X_n \in S_1 \text{ iff } n \equiv 0 \pmod{3}$$

$$X_n \in S_2 \text{ iff } n \equiv 1 \pmod{3}$$

$$X_n \in S_3 \text{ iff } n \equiv 2 \pmod{3}$$

Defn

A recurrent class is called aperiodic iff it is not periodic. Equivalently, there exists a time $n \geq 0$ such that

$$n_{ij}(n) > 0 \text{ for all } i, j \in R$$

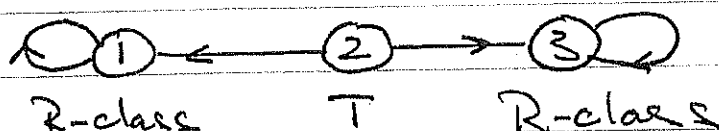
Ex. add a self loop to any state in the previous example.

7.3 steady-state behavior

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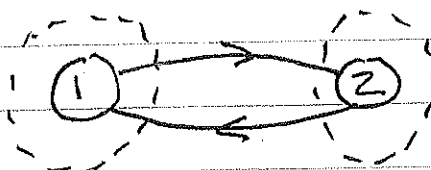
we do not expect the n -step transition probabilities to have limits (not depending on initial state) when there are either

- multiple recurrent classes



or

- periodic recurrent classes



we will write

$$\lim_{n \rightarrow \infty} p_{ij}^{(n)} = \pi_j \quad (\text{for all } i)$$

when the limit exists, and we call π_j the steady-state probability of j .

In the case where these limits exist (independent of i) we interpret

$$\pi_j \approx P(X_n = j) \quad (\text{for } n \text{ large})$$

when do the π_j exist?

Theorem (steady-state convergence)

Consider a Markov chain with a single recurrent class, which is aperiodic.

Then for each $j \in S$:

$$\lim_{n \rightarrow \infty} p_{ij}^{(n)} = \pi_j \quad (\text{for all } i)$$

exists. Furthermore, π_j are solutions to the linear system

$$(1) \quad \sum_{k=1}^m \pi_k p_{kj} = \pi_j \quad (j=1, 2, \dots, m)$$

$$(2) \quad \sum_{k=1}^m \pi_k = 1$$

Also $\pi_j = 0$ if j is transient, and $\pi_j > 0$ if j is recurrent.

Equations (1) are called the balance equations, and follow directly from the Chapman-Kolmogorov equations

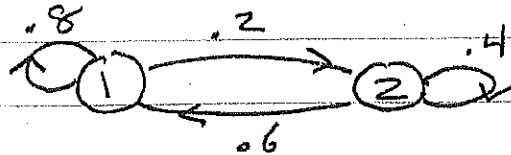
$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1) p_{kj}$$

By taking limits $r_{ij}(n) \rightarrow \pi_j$ and $r_{ik}(n-1) \rightarrow \pi_k$ as $n \rightarrow \infty$.

Equation (2) is called the normalization equation, and follows from the fact that

$$\sum_{j=1}^m P(X_n = j) = 1 \quad (\text{for all } n \geq 0)$$

Ex



$$P_{11} = .8, \quad P_{12} = .2$$

$$P_{21} = .6, \quad P_{22} = .4$$

$$(1) \quad \begin{cases} \pi_1 P_{11} + \pi_2 P_{21} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} = \pi_2 \end{cases}$$

$$(2) \quad \pi_1 + \pi_2 = 1$$

$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 = \pi_1 \Leftrightarrow (.6)\pi_2 = (.2)\pi_1 \\ (.2)\pi_1 + (.4)\pi_2 = \pi_2 \Leftrightarrow (.2)\pi_1 = (.6)\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

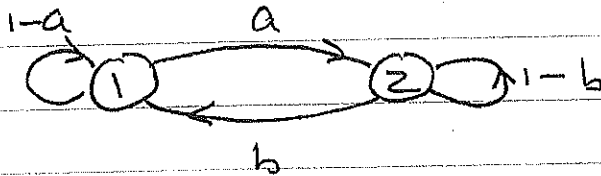
The 1st two give $\pi_1 = 3\pi_2$. The 3rd yields

$$3\pi_2 + \pi_2 = 1 \rightarrow 4\pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}}$$

hence

$$\boxed{\pi_1 = \frac{3}{4}}$$

Ex.



$$\begin{aligned} \pi_1(1-a) + \pi_2 b &= \pi_1 \\ \pi_1 a + \pi_2(1-b) &= \pi_2 \end{aligned}$$

$$\rightarrow \pi_2 b = \pi_1 a$$

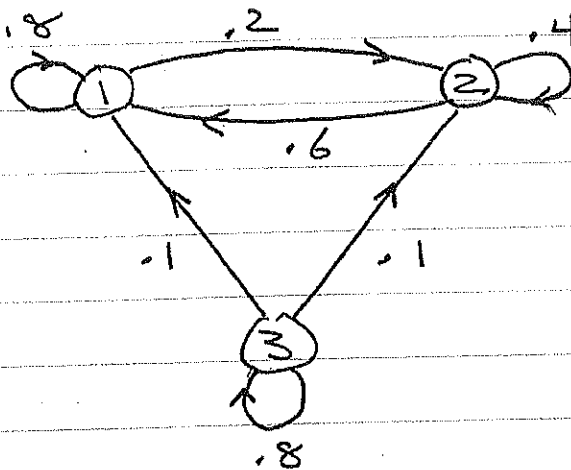
$$\therefore \pi_1 = \frac{b}{a} \pi_2$$

$$\therefore \frac{b}{a} \pi_2 + \pi_2 = 1$$

$$\therefore \pi_2 = \frac{1}{\frac{b}{a} + 1} \rightarrow \boxed{\pi_2 = \frac{a}{a+b}}$$

$$\therefore \boxed{\pi_1 = \frac{b}{a+b}}$$

Ex.



$$\begin{cases} \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} = \pi_2 \\ \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} = \pi_3 \end{cases}$$

$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 + (.1)\pi_3 = \pi_1 \\ (.2)\pi_1 + (.4)\pi_2 + (.1)\pi_3 = \pi_2 \end{cases}$$

$$\begin{cases} -2\pi_1 + 6\pi_2 + \pi_3 = 0 \\ 2\pi_1 - 6\pi_2 + \pi_3 = 0 \end{cases} \rightarrow \begin{cases} 2\pi_3 = 0 \\ \pi_3 = 0 \end{cases}$$

$$\rightarrow \pi_1 = 3\pi_2$$

$$3\pi_2 + \pi_2 = 1 \rightarrow \pi_2 = \frac{1}{4} \therefore \pi_1 = \frac{3}{4}$$

See examples 7.6 and 7.7 on
Pages 355 and 356, resp.

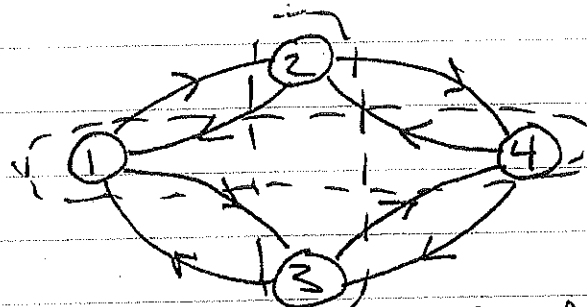
we should have
anticipated this since
we know state 3 is
transient

Periodicity

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Ex.

a). show that the Markov chain model with transition diagram



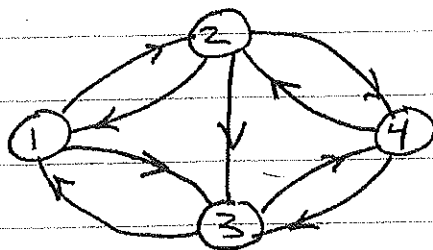
is periodic, and find the period.

Proof

Let $S_0 = \{1, 4\}$ and $S_1 = \{2, 3\}$.

Then all transitions go from S_0 to S_1 , or S_1 to S_0 . Thus the chain is periodic of period 2.

b.) Add a transition $2 \rightarrow 3$ in the above Markov chain model.



show that the resulting chain is not periodic.

Proof

Assume the above chain is Periodic and let S_1 be the set containing state ①. Since the chain has $\checkmark$ transitions from ① to ② and from ① to ③, states ② and ③ must be in the same set.

But the transition ② $\rightarrow$ ③ implies that ② and ③ cannot belong to the same set. This contradiction shows that our assumption was false, and hence the above Markov chain is NOT Periodic. $\blacksquare$

c.) Show that the chain in (b) has steady-state probabilities.

Proof.

observe that $A(1) = A(2) = A(3) = A(4) = \{1, 2, 3, 4\}$, so that the chain contains a single recurrent class. By the steady-state convergence theorem, the limits

$$\pi_j = \lim_{n \rightarrow \infty} P_{ij}(n) \quad (\text{for all } i)$$

exist for $j = 1, 2, 3, 4$. $\blacksquare$

d.) Determine $P(X_{1003}=4 | X_{1000}=1)$
for the chain in (b).

Solution

only one path in the Markov chain goes from ① to ④ in 3 transitions, namely $① \rightarrow ② \rightarrow ③ \rightarrow ④$. Thus

$$P(X_{1003}=4 | X_{1000}=1) = P_{1,4}(3) = \boxed{P_{12} P_{23} P_{34}}.$$