CSE 107 5-8-26

Supplemental Lecture

Ex.

The time T until a new light bulb burns out is Exponential(λ)

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Alice turns on light, leaves and returns t_0 hours later, finding light still on. Thus the event

$$A = \{T > t_0\}$$

has occurred. Let X be the additional time until burnout, i.e.

$$X = T - t_0, \quad T = X + t_0$$

Find the conditional PDF

$$f_{X|A}(x)$$

<u>solution</u>

our plan: ① find cond. CDF

$$F_{X|A}(x) = P(X \le x \mid A)$$

② differentiate

$$\frac{d}{dx} F_{X|A}(x) = f_{X|A}(x)$$

Recall (P.7, 4-30-26) that

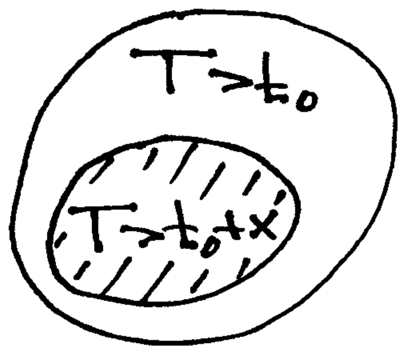
$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \le 0 \end{cases}$$

so for $x \geq 0$:

$$P(X > x \mid A) = P(T > t_0 + x \mid T > t_0)$$

$$= \frac{P(T > t_0 + x,\ T > t_0)}{P(T > t_0)}$$

$$= \frac{P(T > t_0 + x)}{P(T > t_0)}$$



$$= \frac{e^{-\lambda(t_0 + x)}}{e^{-\lambda t_0}}$$

$$= \frac{\cancel{e^{-\lambda t_0}} \cdot e^{-\lambda x}}{\cancel{e^{-\lambda t_0}}}$$

$$= e^{-\lambda x}$$

Thus

$$F_{X|A}(x) = P(X \le x \mid A)$$

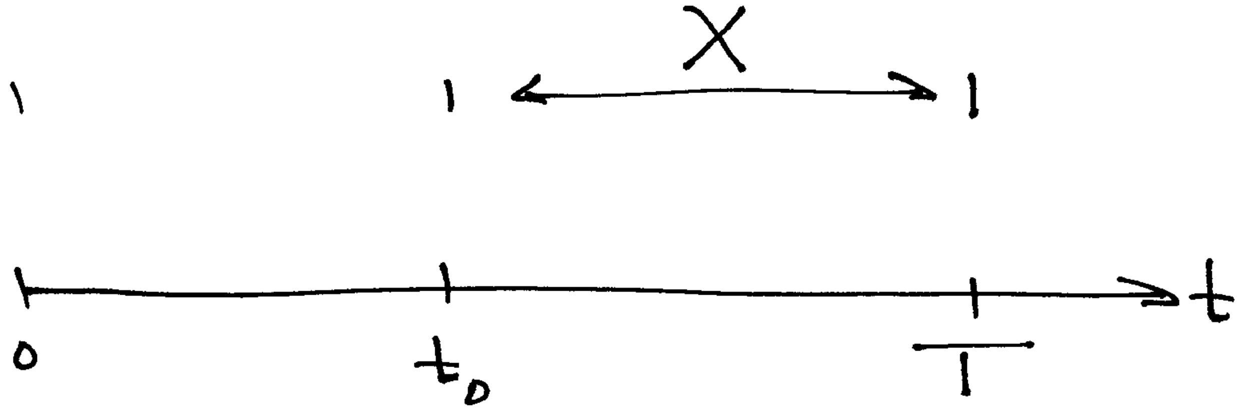
$$= 1 - P(X > x \mid A)$$

$$= \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Hence

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

which is the exponential with parameter λ.



This shows that Exponential(λ) is <u>memoryless</u>.

<u>Exercise</u>

Do same thing for the Geometric(p)

$$\overbrace{\underbrace{0\ 0\ 0\ \ldots\ 0\ 0}_{m}\,\underbrace{0\ 0\ \ldots\ 0\ 0\ 1}_{X}}^{Y}$$

let

$Y = \#$ flips to 1^st^ head

$X = \#$ flips after m^{th} to 1^st^ head

$A = \{Y > m\}$

show that

$$P_{X|A}(k) = P_Y(k) = \begin{cases} p(1-p)^{k-1} & (k=1,2,\dots) \\ 0 & \text{otherwise} \end{cases}$$

Total Probability Theorem for CDF, PDF

If $A_1, A_2, \dots, A_n \subseteq \Omega$ form a partition of Ω, we have

$$P(X \leq x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \leq x \mid A_i)$$

Thus

$$\boxed{F_X(x) = \sum_{i=1}^{n} P(A_i) \cdot F_{X|A_i}(x)}$$

differentiate w.r.t. x gives

$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)}$$

<u>Defn</u>

If X, Y are jointly continuous r.v.s and $f_Y(y) > 0$ for all $y \in \mathbb{R}$, the <u>conditional PDF of X given Y</u> is

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

observe

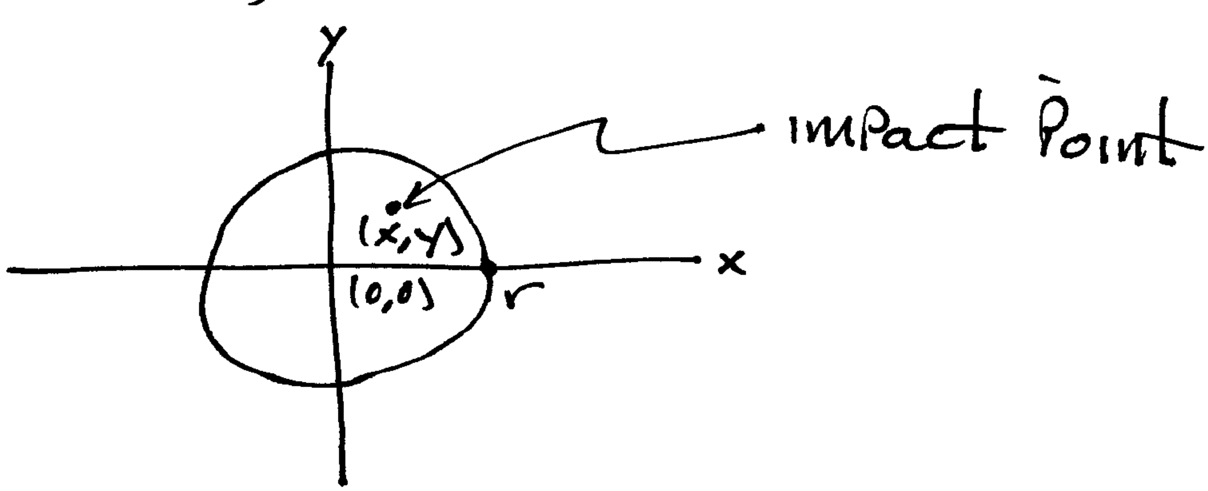
$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \frac{1}{\cancel{f_Y(y)}} \cdot \cancel{f_Y(y)} = 1 .$$

<u>Ex.</u>

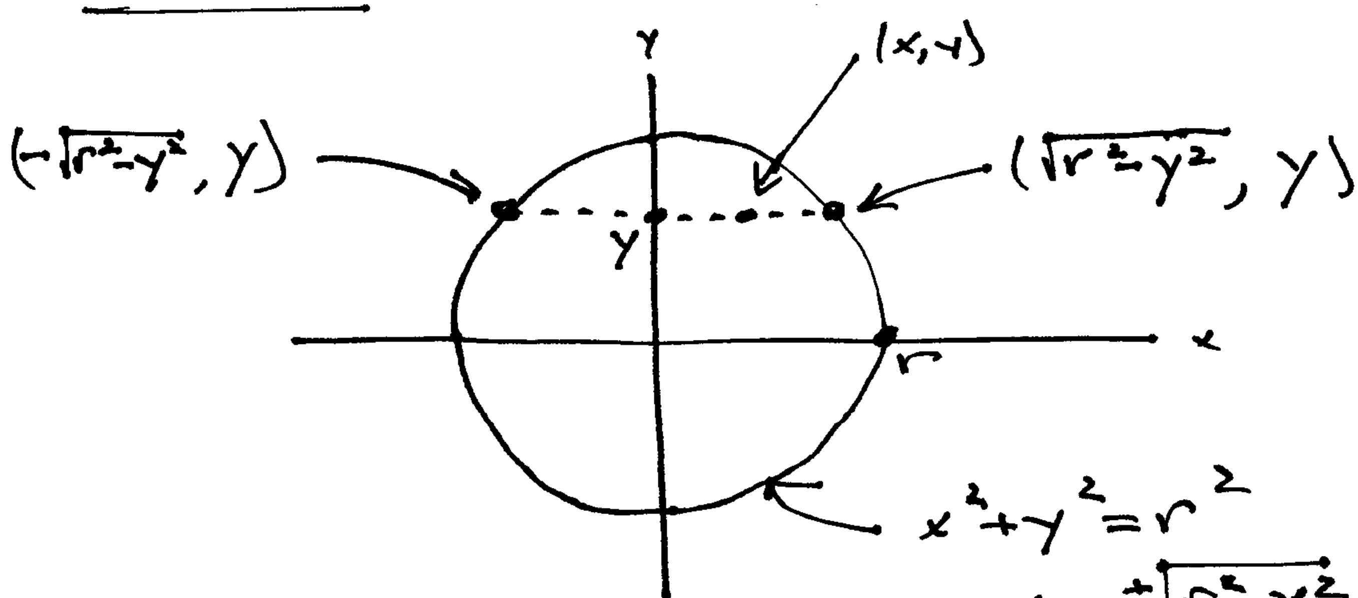
Throw a dart at a circular target of radius r, centered at $(0,0)$



Assume the impact point (x, y) uniformily distributed over the circle. Let (X, Y) be the coordinates of the point.

Find: $f_{X,Y}(x,y)$, $f_Y(y)$ and $f_{X|Y}(x|y)$.

Solution



$x = \pm\sqrt{r^2-y^2}$

$y = \pm\sqrt{r^2-x^2}$

Thus

$$\boxed{f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}}$$

also

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2} \cdot 2\sqrt{r^2-y^2}$$

$$= \boxed{\begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } |y| \le r \ (-r \le y \le r) \\ 0 & \text{otherwise} \end{cases}}$$

finally

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \boxed{\begin{cases} \dfrac{1}{2\sqrt{r^2-y^2}} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}}$$

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<u>We have identities</u> :

- $f_{X|Y}(x|y) f_Y(y) = f_{X,Y}(x,y)$

$$= f_{Y|X}(y|x) \cdot f_X(x)$$

Conditional Expectation

Let X, Y be jointly cont. r.v.s and $A \subseteq \Omega$ with $P(A) > 0$.

Definitions

- $$E[X|A] = \int_{-\infty}^{\infty} x \, f_{X|A}(x) \, dx$$

- $$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$$

Expected Value rules

- $$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) \, f_{X|A}(x) \, dx$$

- $$E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) \cdot f_{X|Y}(x|y) \, dx$$

Total Expectation Theorem

(1) If $A_1, \dots, A_n$ form a Partition of Ω, and $P(A_i) > 0$ (all i), then

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]}$$

(2) if Y is a cont. r.v. then $\{\{Y=y\} \mid y \in \mathbb{R}\}$ forms a Partition of Ω, and

$$\boxed{E[X] = \int_{-\infty}^{\infty} f_Y(y)\, E[X \mid Y=y]\, dy}$$

Proof of (1)

start with total Prob. Thm

$$f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)$$

multiply by x, then integrate w.r.t. x

$$\int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)\, dx$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \int_{-\infty}^{\infty} x f_{X|A_i}(x)\, dx$$

$$= \sum_{i=1}^{n} P(A_i)\, E[X|A_i]$$

$$\therefore E[X] = \sum_{i=1}^{n} P(A_i)\, E[X|A_i] \quad \blacksquare$$

Proof of (2)

start with RHS

$$\int_{-\infty}^{\infty} E[X \mid Y=y]\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \right) \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, \underbrace{f_{X|Y}(x|y) \cdot f_Y(y)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dy\, dx$$

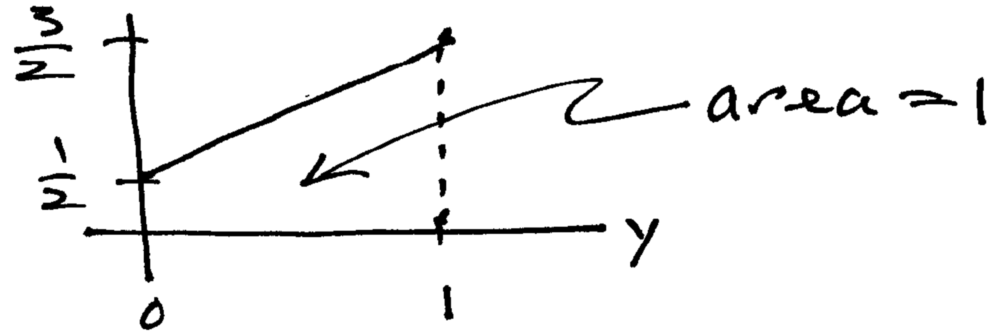
$$= \int_{-\infty}^{\infty} x \underbrace{\left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right)}\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = E[X] \qquad \blacksquare$$

Ex.

let X, Y be jointly cont. with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X]$.

Solution

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

$$= \int_0^1 x\left(\frac{x+y}{y+\frac{1}{2}}\right)dx$$

$$= \frac{1}{y+\frac{1}{2}} \int_0^1 (x^2+yx)\,dx = \frac{1}{y+\frac{1}{2}}\left(\frac{1}{3}x^3+\frac{1}{2}x^2y\right)\Big|_0^1$$

$$= \frac{1}{y+\frac{1}{2}}\left(\frac{1}{3}+\frac{1}{2}y\right)$$

$$= \begin{cases} \dfrac{\frac{1}{2}y+\frac{1}{3}}{y+\frac{1}{2}} & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so by (2)

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y=y] \cdot f_Y(y)\,dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}y + \frac{1}{3}}{\cancel{y+\frac{1}{2}}} \right) \cancel{(y+\tfrac{1}{2})}\,dy$$

$$= \left(\frac{1}{2} \cdot \frac{1}{2} y^2 + \frac{1}{3} y \right) \Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{3} = \frac{3+4}{12} = \boxed{\frac{7}{12}}$$

<u>Exercise</u>

- find $f_{X,Y}(x,y)$
- find $f_X(x)$
- find $E[X]$

Recall

$$E[X \mid Y=y] = \begin{cases} \dfrac{\frac{1}{2}y + \frac{1}{3}}{y+\frac{1}{2}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

This is a fcn. of y.

$$g(y) = E[X \mid Y=y]$$

compose this function with Y:

$$g(Y) = \frac{\frac{1}{2}Y + \frac{1}{3}}{Y + \frac{1}{2}} = E[X \mid Y]$$

so

$$E[g(Y)] = \underbrace{E\big[E[X \mid Y]\big] = E[X]}$$

This is formula (2). also called law of iterated expectations.