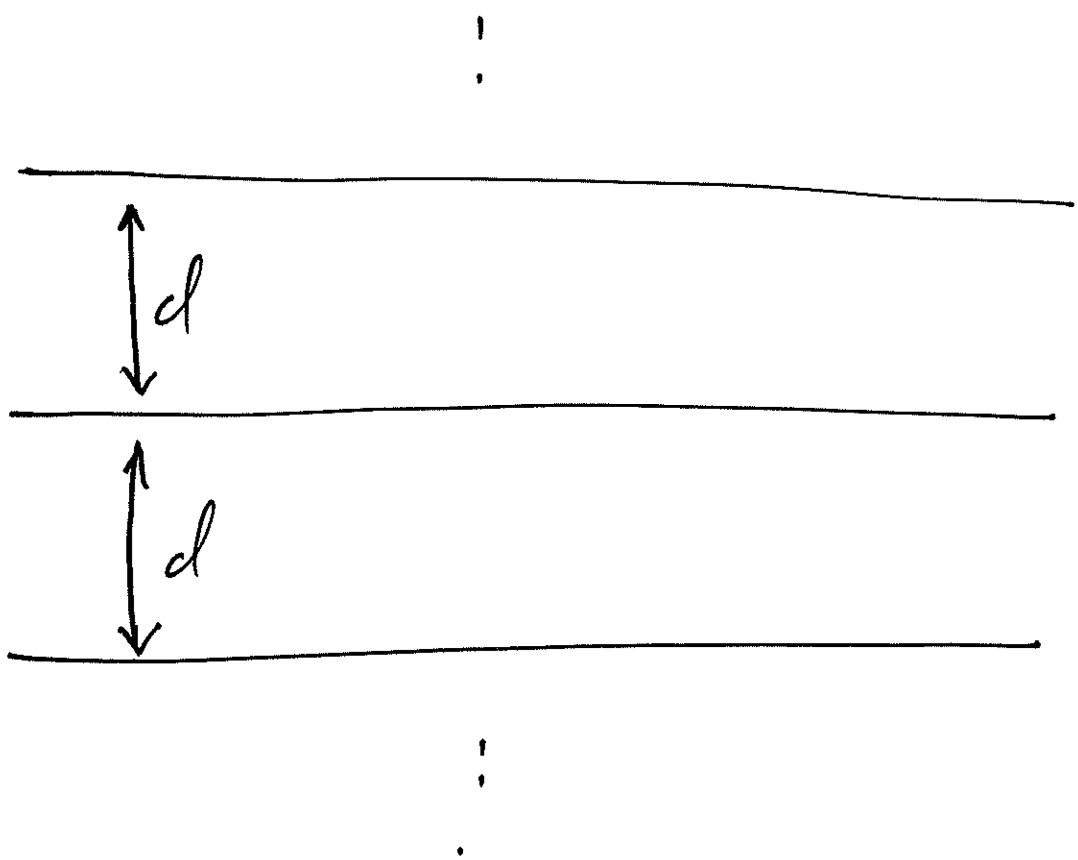
CSE 107 5-7-26

Ex. Buffon's Needle

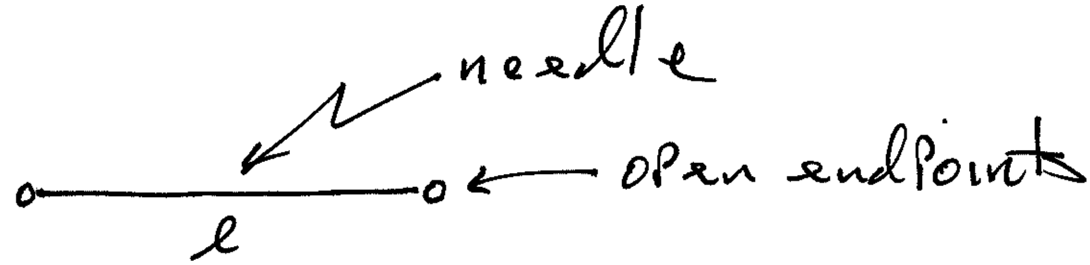
A surface is ruled with parallel lines, d apart.

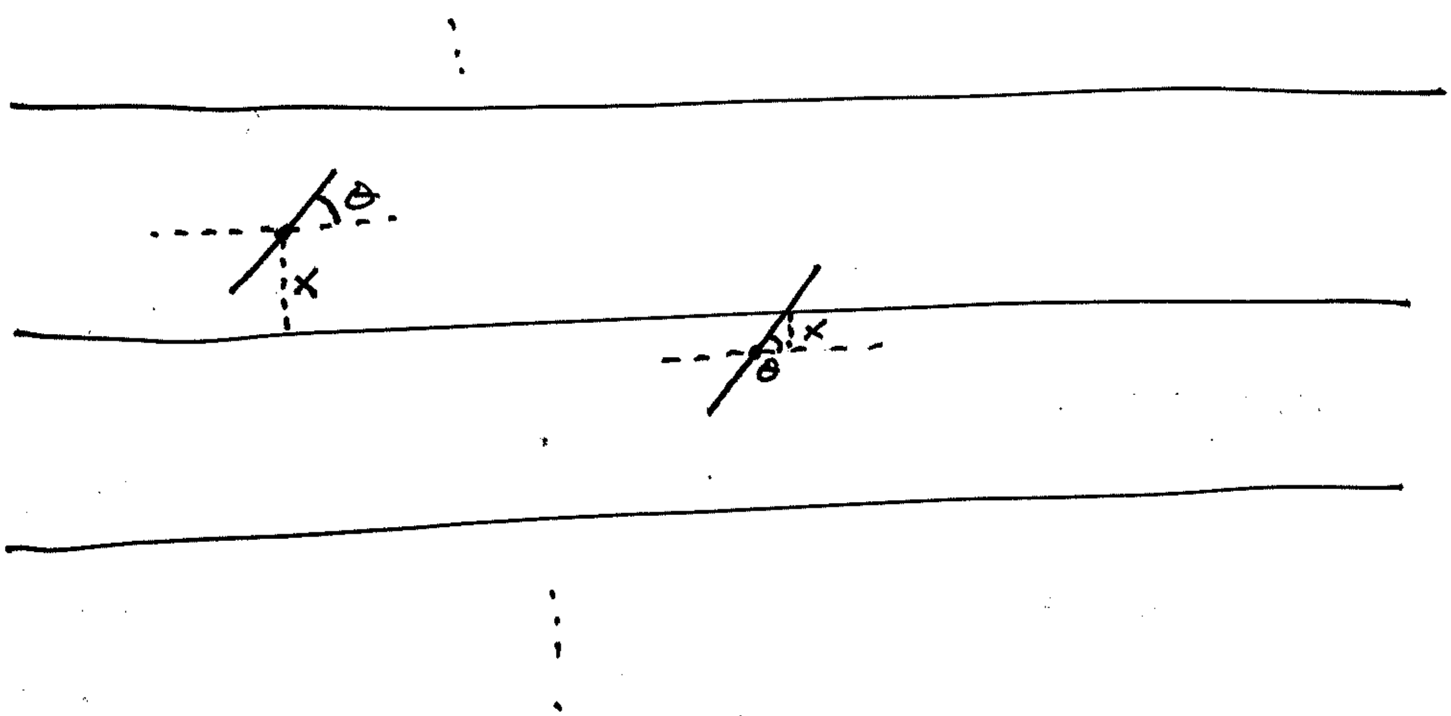


a needle of length l is tossed onto the surface.

What is the probability that the needle intersects one of the lines?

Assume the needle satisfies $l \leq d$, so at most 1 intersection point.

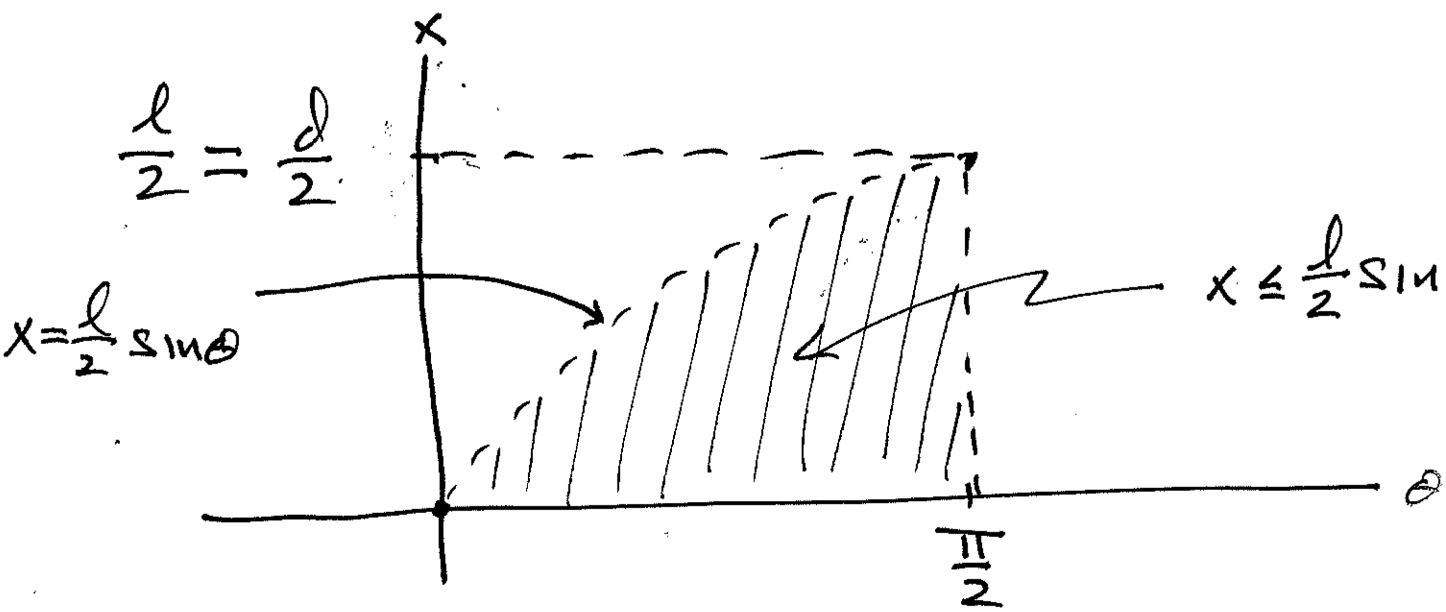


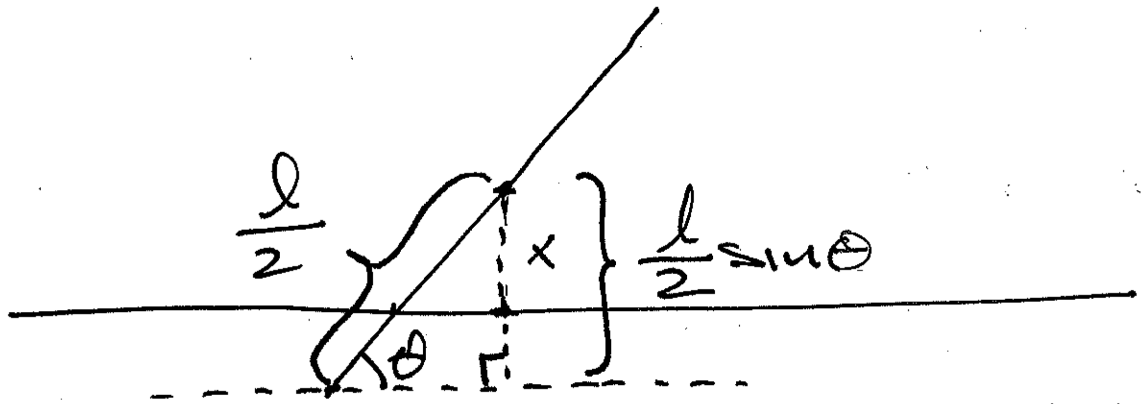


Let X be vertical distance from midpoint of needle to nearest line, and Θ be the acute angle made by the needle and a horizontal.

We model (X, Θ) by a joint uniform PDF on $[0, \frac{d}{2}] \times [0, \frac{\pi}{2}]$.

$$\boxed{f_{X,\Theta}(x,\theta) = \begin{cases} \dfrac{4}{\pi d} & \text{if } 0 \le x \le \frac{d}{2} \text{ and } 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}}$$





<u>note</u>: intersection occurs iff

$$X \le \frac{l}{2}\sin\Theta$$

We seek

$$P\left(X \le \frac{l}{2}\sin\Theta\right)$$

$$= \iint_{x \le \frac{l}{2}\sin\theta} f_{X,\Theta}(x,\theta)\,dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{\frac{l}{2}\sin\theta} dx\,d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} \frac{l}{2} \sin\theta \, d\theta$$

$$= \frac{2l}{\pi d} \left(-\cos\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{2l}{\pi d} \left(-0 - (-1)\right) = \frac{2l}{\pi d}$$

<u>note</u>: if $l = d$, then

$$P(\text{intersect}) = \frac{2}{\pi}$$

$$\therefore \quad \pi = \frac{2}{P(\text{intersect})}$$

approx: Prob of intersect $= \dfrac{\#\text{int}}{\#\text{tosses}}$

$$\therefore \pi \approx \frac{2}{\left(\frac{\#\text{int}}{\#\text{tosses}}\right)} = \boxed{\frac{2 \cdot \#\text{tosses}}{\#\text{intersections}}}$$

approx. to π

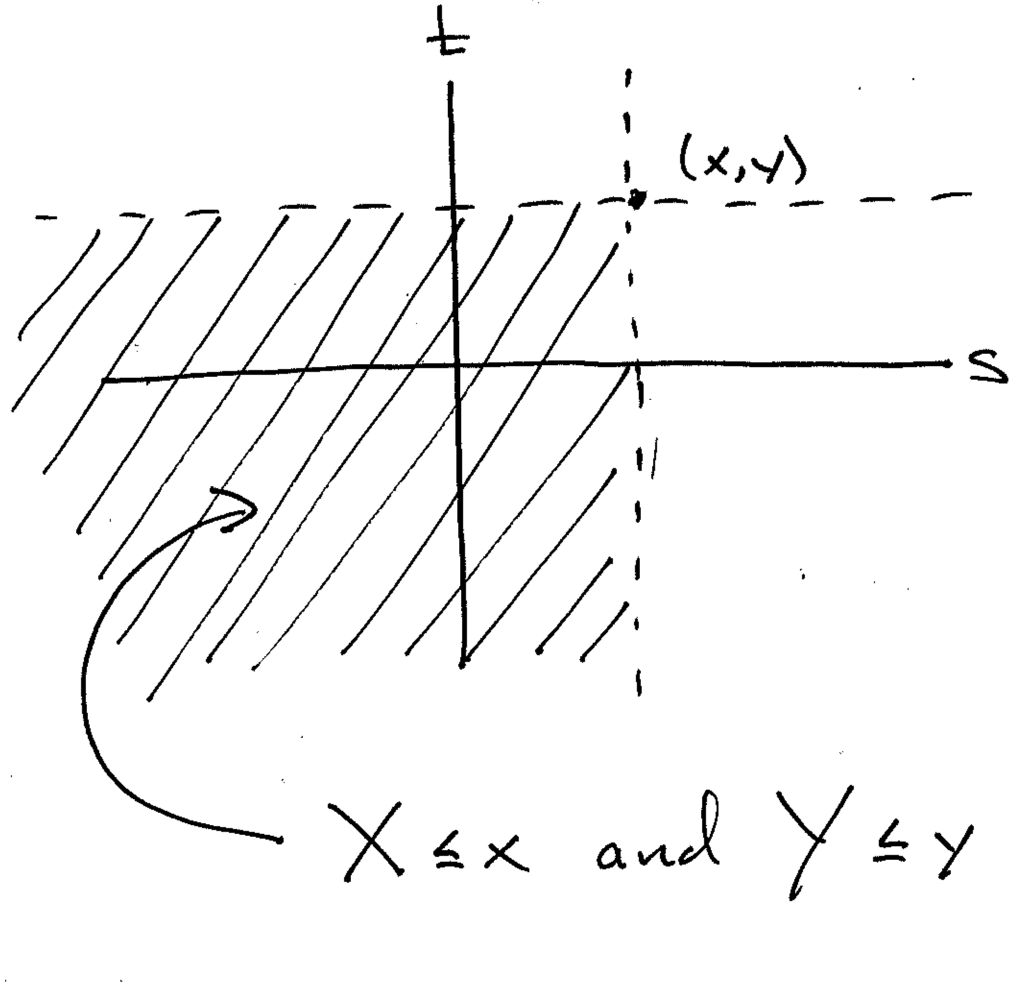
Joint CDFs

If X, Y are two r.v.s, then joint CDF

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

i.e. if X, Y jointly continuous:

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds$$



$X \le x$ and $Y \le y$

We can recover joint PDF by

$$\boxed{f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial y \, \partial x}}$$

Expected Values

If $g : \mathbb{R}^2 \to \mathbb{R}$, we have the 2-variable expected value rule

$$\boxed{E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) \cdot f_{X,Y}(x,y) \, dx \, dy}$$

Special case: $g(x,y) = ax + by + c$

$$E[aX + bY + c]$$

$$= \iint_{\mathbb{R}^2} (ax + by + c) f_{X,Y}(x,y)\, dx\, dy$$

$$= a \iint_{\mathbb{R}^2} x f_{X,Y}(x,y)\, dx\, dy + b \iint_{\mathbb{R}^2} y f_{X,Y}(x,y)\, dx\, dy$$

$$+ c \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\, dx\, dy$$

$$= a E[X] + b E[Y] + c$$

we can do everything for 3, 4, 5, ..., n, of continuous r.v.s

so in particular:

$$E[a_1 X_1 + a_2 X_2 + \dots + a_n X_n + b]$$

$$= a_1 E[X_1] + a_2 E[X_2] + \dots + a_n E[X_n] + b$$

3.5 Conditioning

Let $A \subseteq \Omega$ be an event with $P(A) > 0$.

Defn

The conditional PDF of X given A is a function $f_{X|A}(x)$ satisfying

$$P(X \in I \mid A) = \int_I f_{X|A}(x)\,dx$$

for 'any' subset $I \subseteq \mathbb{R}$.

It is required that

$$\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = P(\Omega|A) = 1$$

<u>Ex.</u>

The time T until a new light bulb burns out is Exponential(λ)

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Alice turns on the light and returns t_0 hours later, finding light still on.

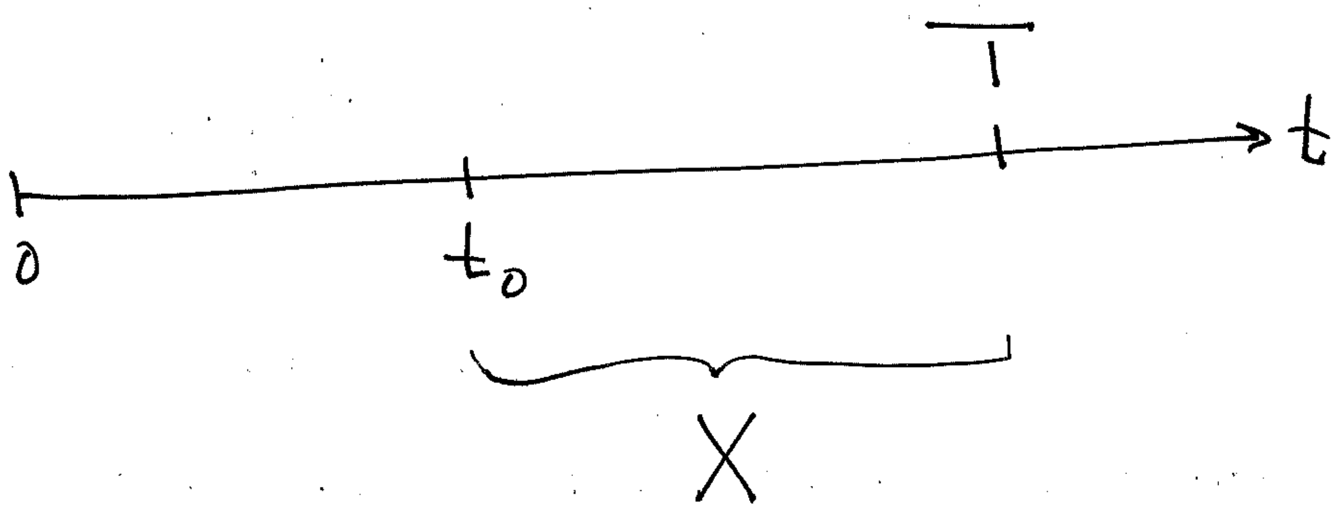
Thus the event

$$\{T > t_0\} = A$$

has occurred. Let X be the additional time to burnout

$$\therefore T = X + t_0$$

$$X = T - t_0$$



Find: $f_{X|A}(x)$

solution

1st find conditional CDF defined as

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= \int_{-\infty}^{x} f_{X|A}(s)\, ds$$

then differentiate to get

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x)$$

Recall from P.7 on notes 4-30-26

that

$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

$\vdots$