- standard Normal: $\mu=0, \sigma=1$
- notation for CDF of std. normal Y is:

$$\Phi(y) = P(Y \le y)$$
$$= P(Y < y)$$
$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} \, dt$$

- see standard Normal CDF table

- Symmetry of $\Phi(y)$

$$\Phi(-y) = 1 - \Phi(y)$$

- If $0 \le y \le 3.49$, look up in table
- If $y > 3.49$, $\Phi(y) \approx 1$
- If $y < 0$, use symmetry

Ex

- $\Phi(1.84) = .9671$
- $\Phi(-1.84) = 1 - \Phi(1.84) = .0329$

Let $X \sim \text{Normal}(\mu, \sigma)$,
then let

$$Y = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

Then Y is normal with

$$E[Y] = \frac{1}{\sigma}E[X] - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$

and

$$Var(Y) = \frac{1}{\sigma^2}Var(X) = \frac{1}{\sigma^2}\cdot\sigma^2 = 1$$

$\therefore$ Y is standard Normal.

Thus

$$F_X(x) = P(X \le x)$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Y \le \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Ex.

The mean high temp. in Santa Cruz on May 5 is 66° F, with std. dev. 4.4°

what is the probability that the high temp. will be $\leq 57^\circ$.

Solution

Let X the high temp.

Then

$$P(X \le 57) = P\Big(\underbrace{\frac{X-66}{4.4}}_{Y} \le \frac{57-66}{4.4}\Big)$$

$$= P(Y \le -2.04)$$

$$= \Phi(-2.04)$$

$$= 1 - \Phi(2.04)$$

$$= 1 - 0.9793$$

$$= \boxed{.0207}$$

Ex.

What is the Prob. that a normal r.v. X is within 1 standard deviation of its mean?

Solution

Let X be Normal(μ, σ).

$$P(\mu-\sigma \le X \le \mu+\sigma)$$

$$= P\left(\frac{(\mu-\sigma)-\cancel{\mu}}{\sigma} \le \underbrace{\frac{X-\mu}{\sigma}}_{Y} \le \frac{(\cancel{\mu}+\sigma)-\cancel{\mu}}{\sigma}\right)$$

$$= P(-1 \le Y \le 1)$$

$$= P(Y \leq 1) - P(Y < -1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$= 2(.8413) - 1$$

$$= \boxed{0.6826}$$

Similarly, the Prob. of being within 2σ of μ is

$$= 2\Phi(2)-1 = 2(.9772)-1 = \boxed{.9544}$$

and for 3σ:

$$= 2\Phi(3)-1 = 2(.9987)-1 = \boxed{.9974}$$

Error Function

$$\mathrm{erf}(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

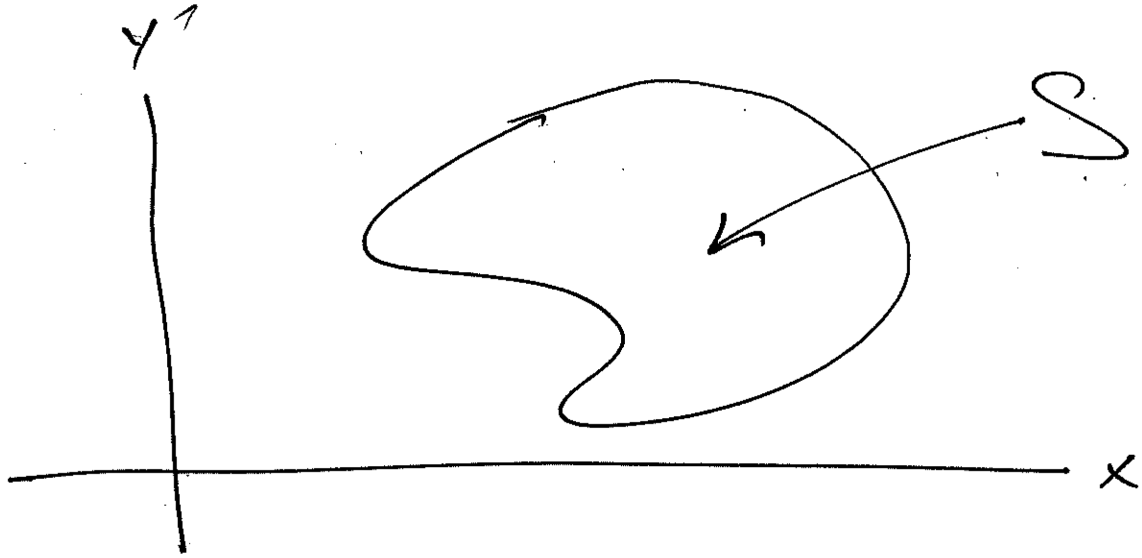
3.4 Joint PDF of multiple r.v.s

Defn

We say r.v.s X, Y are **Jointly Continuous** iff there exists a function $f_{X,Y}(x,y)$ such that

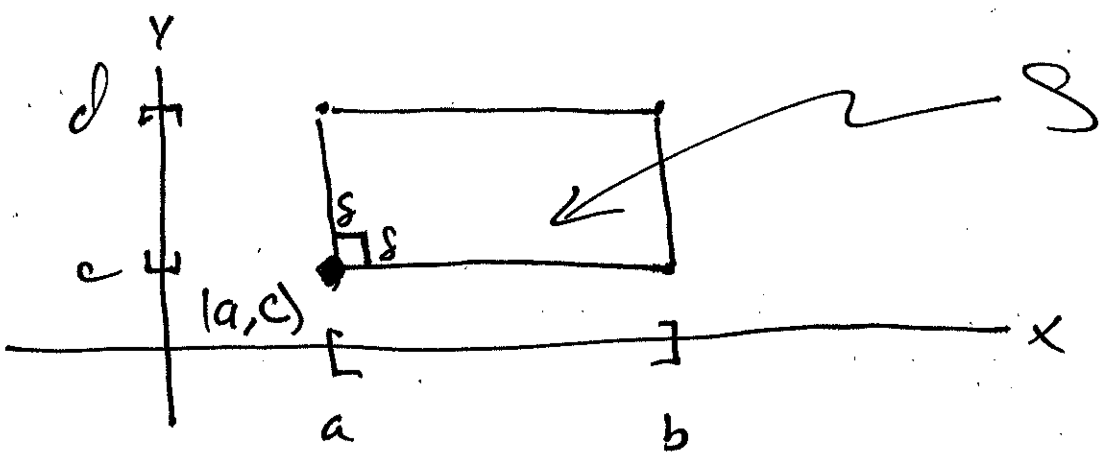
$$P((X,Y) \in S) = \iint_S f_{X,Y}(x,y)\,dx\,dy$$

for 'any' $S \subseteq \mathbb{R}^2$. The fcn $f_{X,Y}(x,y)$ the joint PDF of X, Y.



In Particular, if $S=[a,b]\times[c,d]$

$=\{(x,y) \mid a\le x\le b \text{ and } c\le y\le d\}$, then



$$P(a\le X\le b,\ c\le Y\le d)=\int_a^b\int_c^d f_{X,Y}(x,y)\,dy\,dx$$

If $S = \mathbb{R}^2$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = P(\Omega) = 1$$

If $\delta > 0$ is small, and $f_{X,Y}(x,y)$ is continuous near (a,c), then

$$\overbrace{P(a \le X \le a+\delta,\ c \le Y \le c+\delta)}^{\text{Probability}}$$

$$= \int_a^{a+\delta}\int_c^{c+\delta} f_{X,Y}(x,y)\,dy\,dx$$

$$\approx f_{X,Y}(a,c)\cdot \underbrace{\delta^2}_{\substack{\text{units}\\ \text{area}}}$$

Thus $f_{X,Y}(a,c)$ has units

Probability/area, i.e. <u>density</u>.

Recall if $I \subseteq \mathbb{R}$, we have

$$P(X \in I) = \int_I f_X(x)\,dx$$

also

$$P(X \in I) = P(X \in I, \Omega)$$

$$= P(X \in I, Y \in \mathbb{R})$$

$$= \iint_{I \times \mathbb{R}} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_I \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \right) dx$$

Therefore

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy} \quad \text{(marginal)}$$

similarly

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx} \quad \text{(marginal)}$$

observe

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \right) dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dy\,dx$$

also

$$E[Y] = \iint_{\mathbb{R}^2} y f_{X,Y}(x,y)\,dx\,dy$$

Ex. Joint Uniform continuous on

$R = [a,b] \times [c,d]$,

$$f_{X,Y}(x,y) = \begin{cases} \alpha & (x,y) \in R \\ 0 & (x,y) \notin R \end{cases}$$

we must have

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \iint_R \alpha \, dx\,dy$$

$$= \alpha \cdot \iint_R dx\,dy = \alpha \cdot \text{area}(R)$$

$$\therefore \quad \alpha = \frac{1}{\text{area}(R)} = \frac{1}{(b-a)(d-c)}.$$