CSE 107 5-29-26

Supplemental Lecture

⋮

The approximation in (e) refers to the limit as $\delta \to 0$.

$$h(k,\delta) = \begin{cases} 1-\lambda\delta & k=0 \\ \lambda\delta & k=1 \\ 0 & k=2,3,\ldots \end{cases} + o(\delta)$$

Here $o(\delta)$ stands for a function with the property

$$\lim_{\delta \to 0} \left(\frac{o(\delta)}{\delta} \right) = 0$$

For instance $c\delta^2$ has this property. Also $(a_2\delta^2 + a_3\delta^3 + a_4\delta^4 + \cdots)$ has this property.

Observe that

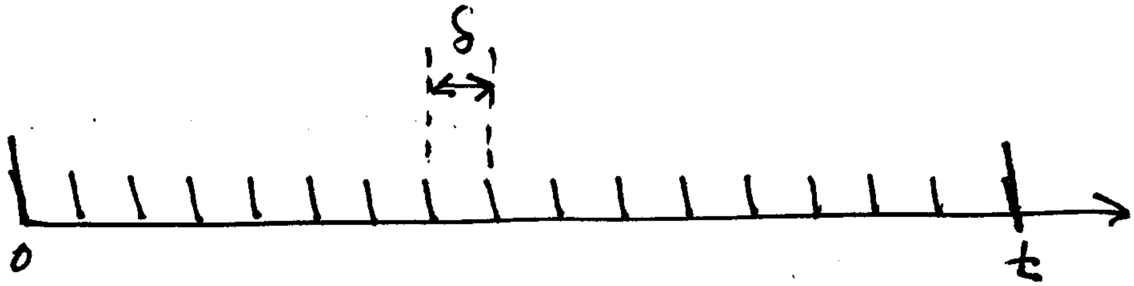
$$E[\#\text{ arrivals in } [t_0, t_0+\delta]]$$

$$= 0\cdot(1-\lambda\delta) + 1\cdot\lambda\delta + 0$$

$$= \lambda\delta$$

Thus λ is the average # arrivals per unit of time. Hence λ is the <u>arrival rate</u> of the Process.

To reason about the # arrivals in a <u>large</u> interval $[0, t]$ (or the interval $[t_0, t_0+t]$), we divide it into many small intervals of width $\delta > 0$.



Let $n = \frac{t}{\delta}$ be the # of subintervals

choose $\delta = \frac{t}{n}$ so small (i.e. n so large)

that Prob. of more than 1 arrival

in a sub-interval is negligible.

We have (approximately) a Bernoulli

Process with Prob. of arrival

$$p = \lambda\delta = \frac{\lambda t}{n}$$

The approx. improves as $\delta \to 0$ ($n \to \infty$).

The # of arrivals of a Bernoulli

Process is Binomial(n, p). So

approximately

$P(k \text{ arrivals in } [0, t])$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \left(\frac{\lambda t}{n}\right)\right)^{n-k} \longrightarrow ??$$

In the limit $n \to \infty$ (same as $\delta \to$)
we have

$P(k \text{ arrivals in } [0, t])$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!} \qquad (*)$$

Let N_t denote the # of arrivals in $[0, t]$ by the Poisson Process.

Thus

$$P_{N_t}(k) = P(N_t = k)$$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

and Therefore

$$\boxed{N_t \sim \text{Poisson}(\lambda t)}$$

(*) if a homogeneous, continuous time arrival Process has $N_t \sim \text{Poisson}(\lambda t)$ then it satisfies the small interval Probabilities (c)

We also know $E[N_t] = \lambda t$ and $Var(N_t) = \lambda t$.

Let T be the time to 1^{st} arrival in a Poisson Process.

<u>Goal</u>: find distribution (PDF) of T

observe that (if $t \geq 0$)

$$\{T > t\} = \{N_t = 0\}$$

Thus

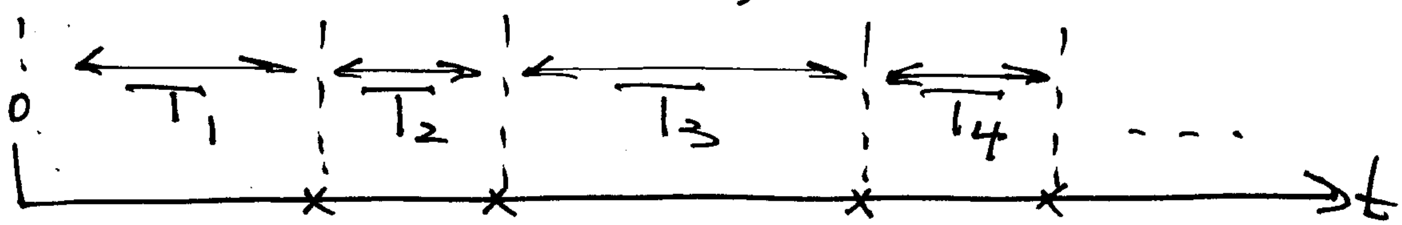
$$\begin{aligned} F_T(t) &= P(T \leq t) \\ &= 1 - P(T > t) \\ &= 1 - P(N_t = 0) \\ &= 1 - p_{N_t}(0) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

and upon differentiating w.r.t. t

$$f_{T_1}(t) = \lambda e^{-\lambda t} \quad (t \geq 0)$$

Thus $T_1 \sim \text{Exponential}(\lambda)$

Let T_k be the k^{th} inter-arrival time $(k = 1, 2, \ldots)$



After each arrival, the Poisson Process has a fresh start. Hence

$$\boxed{T_k \sim \text{Exponential}(\lambda)}$$

$$\therefore \quad f_{T_k}(t) = \lambda e^{-\lambda t} \quad (t \geq 0)$$

Now define

$$Y_K = T_1 + T_2 + \cdots + T_K \quad (K = 1, 2, \ldots)$$

i.e. Y_K is the time to K^{th} arrival. The distribution of Y_K is derived in text (P. 316-317)

Its called <u>Erlang of order K</u>

$$f_{Y_K}(t) = \frac{\lambda^K t^{K-1} e^{-\lambda t}}{(K-1)!} \quad (t \geq 0)$$

i.e.

$$\boxed{Y_K \sim \text{Erlang}(K, \lambda)}$$

Exercise

Prove that the sum of k independent Exponential(λ) r.v.s is Erlang(k, λ).

Hint:

use induction on k. when $k=1$ Erlang(1, λ) $\sim$ Exponential(λ). Use convolution on induction step to show

$$\underbrace{Y_k}_{\text{Erlang}(k,\lambda)} = \underbrace{Y_{k-1}}_{\text{Erlang}(k-1,\lambda)} + \underbrace{T_k}_{\text{Exponential}(\lambda)}$$

Summary: Bernoulli vs. Poisson

	Bernoulli	Poisson
time:	$t \in \{1, 2, \dots\}$	$t \in [0, \infty)$
rate:	p/trial	λ/unit time
N_t:	Binomial(t, p)	Poisson(λt)
T_k:	Geometric(p)	Exponential(λ)
Y_k:	Pascal(k, p)	Erlang(k, λ)
X_t:	Bernoulli(p)	$\underbrace{?}_{N_{t+dt} - N_t}$

<u>note</u>: $X_t = N_t - N_{t-1}$

$N_t = X_1 + \dots + X_t$

Ex

You receive email in a Poisson Process at rate $\lambda = 2$ msgs/hour

You currently have no messages.

a.) What is Prob. of 3 new msgs. in next 30 minutes?

$$P(N_{\frac{1}{2}} = 3) = P_{N_{\frac{1}{2}}}(3)$$

$$= e^{-2 \cdot \frac{1}{2}} \cdot \frac{(2 \cdot \frac{1}{2})^3}{3!}$$

$$= e^{-1} \cdot \frac{1}{6} = \boxed{.0613}$$

b.) What is the Prob. of no msgs in 30 min?

$$P(N_{\frac{1}{2}} = 0) = P_{N_{\frac{1}{2}}}(0) = e^{-2 \cdot \frac{1}{2}} \cdot \frac{(2 \cdot \frac{1}{2})^0}{0!}$$

$$= e^{-1} = \boxed{.3679}$$

c.) What's the expected time of 5^{th} new message?

$$E[Y_5] = E[T_1 + \cdots + T_5]$$

$$= 5 \cdot E[T]$$

$$= 5 \cdot \frac{1}{\lambda} = \frac{5}{2} = \boxed{2.5 \text{ hours}}$$

The sum of Poisson r.v.s

Let X, Y be indep. Poisson r.v.s with param λ_1, λ_2, resp.

Theorem

$Z = X + Y$ is Poisson$(\lambda_1 + \lambda_2)$.

We have

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \qquad (k = 0, 1, \ldots)$$

and

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \qquad (k = 0, 1, \ldots)$$

Thus

$$P_Z(n) = \sum_{k=0}^{\infty} P_X(k)\, P_Y(n-k)$$

$\boxed{\begin{array}{c} \text{require:} \\ k \geq 0 \\ n-k \geq 0 \\ \therefore\ 0 \leq k \leq n \end{array}}$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \lambda_1^k \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot \lambda_1^k \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot (\lambda_1+\lambda_2)^n$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!}$$

Thus $Z \sim \text{Poisson}(\lambda_1+\lambda_2)$. ∎

<u>Ex.</u>

During Rush hour (8-9 am), accidents occur as a Poisson Process with rate $\lambda_1 = 5$/hour. During time 9-11 am, accidents are also a Poisson Process with rate $\lambda_2 = 3$/hour. find PMF of total # of accidents 8-11 am?

Solution

Let

$$N_1 = \#\text{accidents in } \overbrace{8-9}^{\text{len}=1} \text{ am}$$

$$N_2 = \text{'' \quad '' \quad ''} \quad \underbrace{9-11}_{\text{len}=2} \text{ am}$$

and

$$N = N_1 + N_2 = \#\text{accidents in } 8\text{-}11 \text{ am}.$$

We know

$$N_1 \sim \text{Poisson}(\overbrace{5 \cdot 1}^{5})$$

$$N_2 \sim \text{Poisson}(\underbrace{3 \cdot 2}_{6})$$

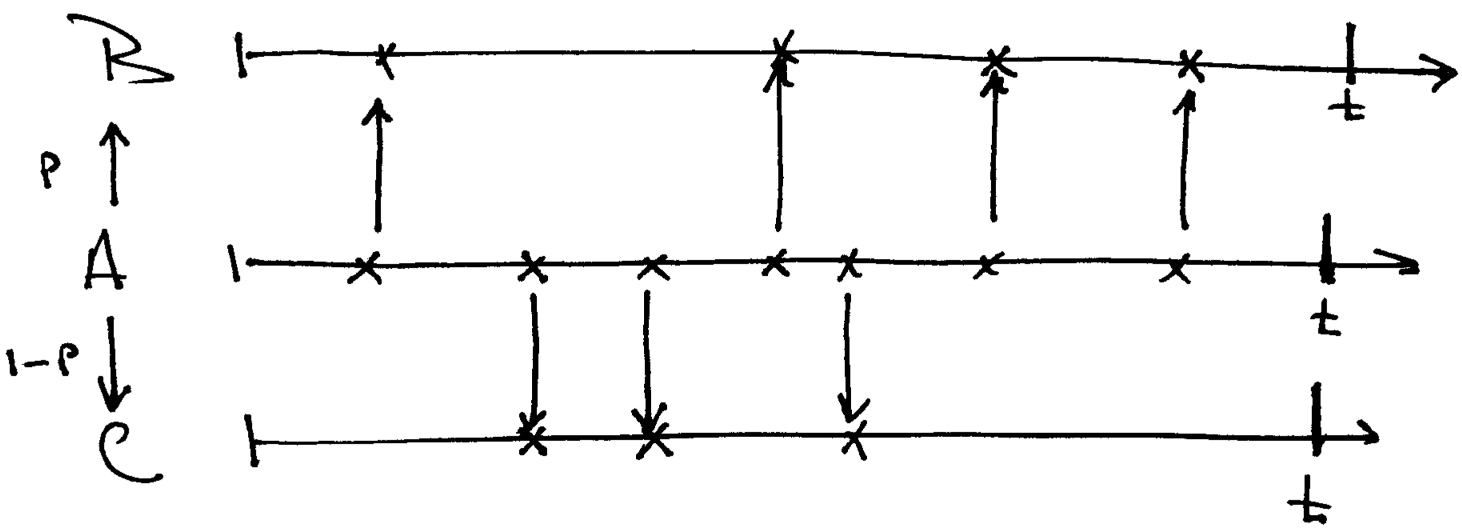
Thus

$$N \sim \text{Poisson}(\overbrace{5+6}^{11})$$

$$\therefore \boxed{p_N(n) = e^{-11} \cdot \frac{11^n}{n!} \qquad (n = 0, 1, 2, \ldots)}$$

Splitting a Poisson Process

Let A be a Poisson Process, rate $=\lambda$.
Split A into 2 arrival Processes B and C with Prob. p, $1-p$, resp..



Let

$N = \#$ arrivals of A in $[0,t]$

$X = \ldots\ldots$ B $\ldots$

$Y = \ldots\ldots$ C $\ldots$

What kind of Processes are B, C ?

We know $N \sim \text{Poisson}(\lambda t)$.

$$P_N(n) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \quad (n = 0, 1, 2, \ldots)$$

so

$$P_X(k) = P(X = k)$$

$$= \sum_{n=k}^{\infty} P(X = k \mid N = n) \cdot P(N = n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} \cdot p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} \cdot p^k (1-p)^{n-k} e^{-\lambda t} \cdot \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^k}{k!} \cdot e^{-\lambda t} \cdot \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \cdot (\lambda t)^n}{(n-k)!}$$

$$\left\{ \begin{array}{l} \text{substitute: } n \leftarrow n+k \\ \text{so } n = k \rightarrow n+k = k \Rightarrow n = 0 \end{array} \right.$$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n (\lambda t)^{n+k}}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p)\cdot \lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} \cdot e^{-\lambda t} \cdot e^{(1-p)\lambda t}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\cancel{\lambda t} + \cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!}$$

Thus $X \sim \text{Poisson}((p\lambda)t)$. We see B is a Poisson Process with rate: $\boxed{p\lambda}$

similarly C is a P.P. of rate: $\boxed{(1-p)\lambda}$.

See Ex. 6.13 on P. 318

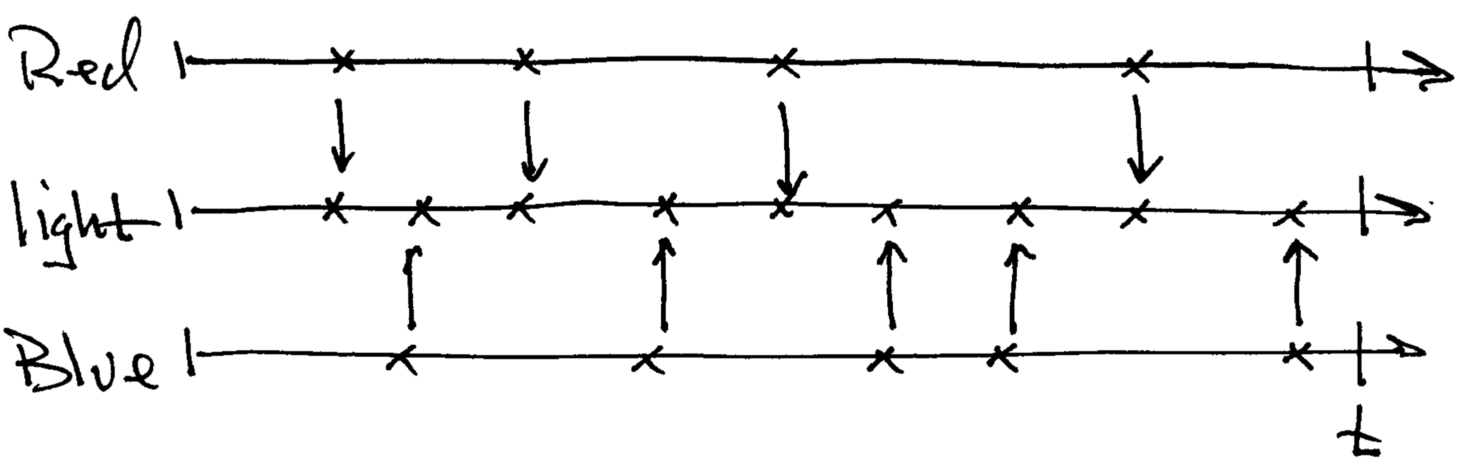
Merging Poisson Processes

Ex

An observer sees 2 lights flash at random times: Red / Blue.

$$\begin{cases} \text{Red: Poisson Process: rate} = \lambda_1 \\ \text{Blue: } \quad " \quad\quad " \quad \text{: rate} = \lambda_2 \end{cases}$$

The observer is color blind.
What kind of Process does observer see?



Let

X = # Red lights in $[0,t]$

Y = # Blue " in $[0,t]$

M = # lights in $[0,t]$.

Know $X \sim \text{Poisson}(\lambda_1 \cdot t)$

$Y \sim \text{Poisson}(\lambda_2 \cdot t)$

Thus $M = X + Y$ is necessarily $\text{Poisson}(\lambda_1 t + \lambda_2 t) = \text{Poisson}((\lambda_1 + \lambda_2) t)$

$\therefore$ M is a Poisson Process with rate $= \lambda_1 + \lambda_2$.

Ex.

suppose observer sees a single flash. What is the Prob. it was Red?

Solution 1 (uses small interval Prob.)

Let $\delta > 0$ be the width of a small interval containing only that flash of light. We seek

$$P(1 \text{ Red} \mid 1 \text{ light})$$

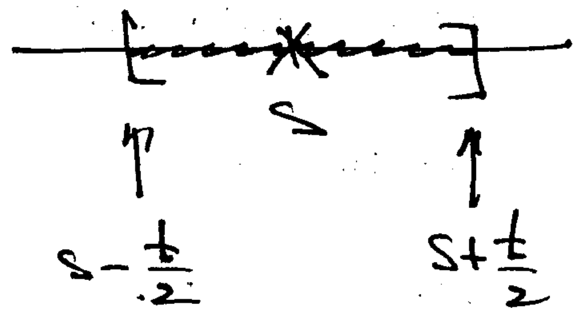
$$= \frac{P(1 \text{ Red}, 1 \text{ light})}{P(1 \text{ light})}$$

$$= \frac{P(1 \text{ Red})}{P(1 \text{ light})} = \frac{\lambda_1 \cancel{\delta}}{(\lambda_1+\lambda_2)\cancel{\delta}} = \boxed{\frac{\lambda_1}{\lambda_1+\lambda_2}}$$

Solution 2

Suppose light flashes at time s.
Consider an interval of length t centered at s:

$$I_t = \left[s - \frac{t}{2}, s + \frac{t}{2} \right]$$



Then

$$P(1 \text{ Red in } I_t \mid 1 \text{ light in } I_t)$$

$$= \frac{P(1 \text{ Red in } I_t, 1 \text{ light in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \dfrac{(\lambda_1 t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \dfrac{((\lambda_1+\lambda_2)t)^1}{1!}}$$

$$= e^{\lambda_2 t} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} \xrightarrow[t \to 0]{\text{as}} \boxed{\frac{\lambda_1}{\lambda_1+\lambda_2}}$$

$\blacksquare$