CSE 107 5-26-26

6.1 The Bernoulli Process

Defn

The Bernoulli Process is a sequence

$$X_1, X_2, \ldots\ldots$$

of independent Bernoulli(p) r.v.s

Two basic questions

① Given $n \geq 0$, how many succeses occur from $t=1$ to $t=n$?

answer: Binomial(n, p)

(2) how many trials until the 1^{st} success?

<u>answer</u>: Geometric(p)

<u>Memorylessness</u> - fresh start.

Let T be the time (i.e. number of trials) until the 1^{st} success. Then

$$P_T(t) = (1-p)^{t-1} \cdot p \quad (t = 1, 2, \ldots)$$

we know $E[T] = \frac{1}{p}$ and $\mathrm{Var}(T) = \frac{1-p}{p^2}$

Suppose we observe

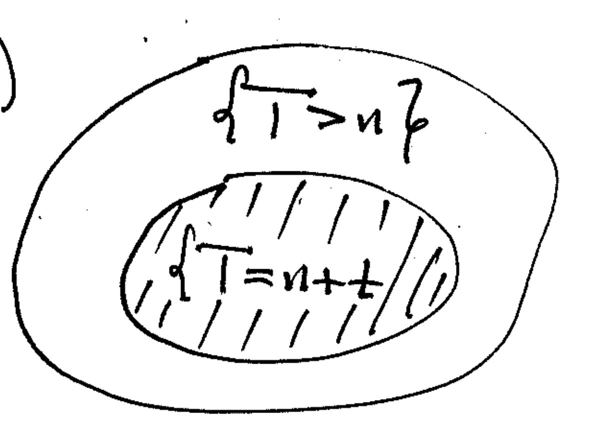
$$X_1, X_2, \ldots, X_n$$

are all failures. Then $\{T>n\}$ has occurred. What's the (conditional) distribution (PMF) of $T-n$, the remaining time to 1st head?

$$p_{T-n|T>n}(t) = P(T-n=t \mid T>n)$$

$$= \frac{P(T=n+t,\ T>n)}{P(T>n)}$$

$$= \frac{P(T=n+t)}{P(T>n)}$$



$$= \frac{(1-p)^{n+t-1} \cdot p}{(1-p)^n}$$

$$= (1-p)^{t-1} \cdot p$$

$$= P(T = t)$$

$$= P_T(t)$$

Interarrival times $\left\{ \begin{array}{l} \text{note:} \\ \text{success} \\ = \text{arrival} \end{array} \right.$

Let

T_1 = time to 1^{st} arrival

T_2 = time from T_1 to 2^{nd} arrival

T_3 = " " T_2 " 3^{rd} arrival

⋮

T_k = time from T_{k-1} to k^{th} arrival

⋮

Because of the 'fresh start' Property, each $T_i \sim$ Geometric(p)

{all T_i are indep.}

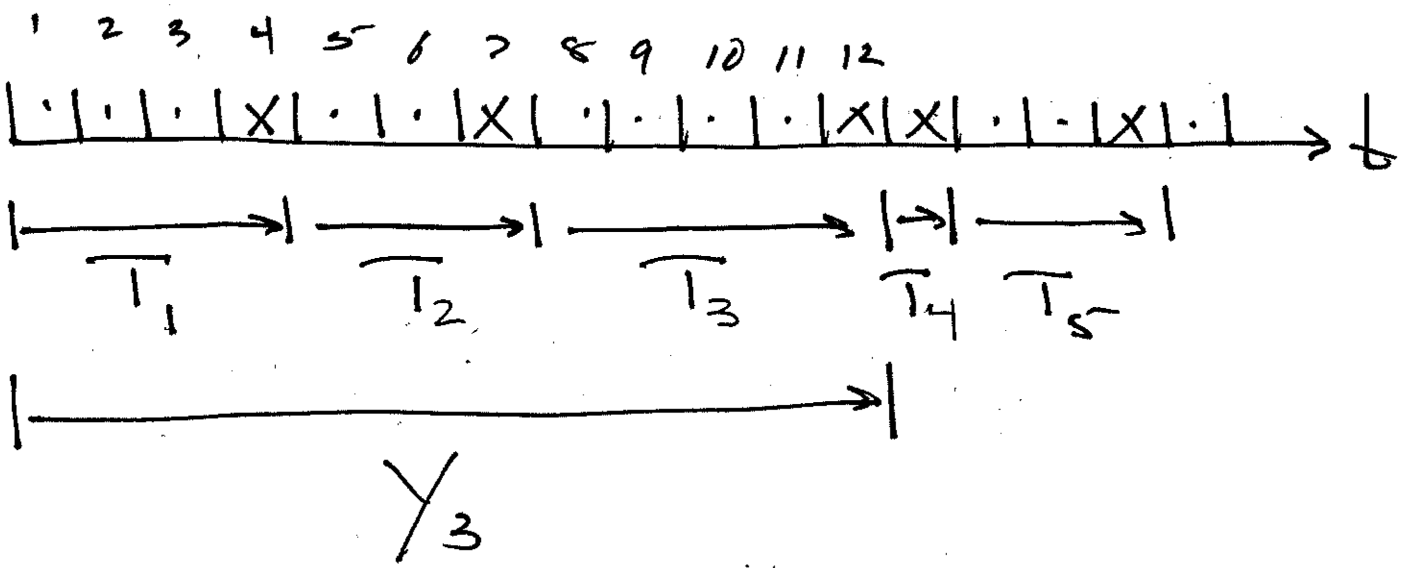
Let

$$Y_k = T_1 + T_2 + \cdots + T_k$$

$$= \text{time to } k^{th} \text{ arrival}$$

Alternately can define

$$\begin{cases} T_1 = Y_1 \\ T_K = Y_K - Y_{K-1} \quad (K = 2, 3, \ldots) \end{cases}$$



Exercise

- find $Y_1 = T_1$
- find $Y_2 = Y_1 + T_2$ (use convolution)
- find $Y_3 = Y_2 + T_3$ (" ")
- find $Y_4 = Y_3 + T_4$ (" ")

$\vdots$

- find $Y_k = Y_{k-1} + T_k$

$\vdots$

Prove general formula using induction on k.

Another Proof

$$P_{Y_K}(t) = P(Y_K = t)$$

$= P\big((K-1)$ arrivals in $[1, t-1]$, **and** 1 arrival at time $t\big)$

$= \underbrace{P(K-1 \text{ arriv. in } [1,t-1])}_{\text{Binomial}(t-1,\, p;\, K-1)} \cdot \underbrace{P(\text{arriv. at } t)}_{p}$

by independence

$$= \binom{t-1}{K-1} \cdot p^{K-1} \cdot (1-p)^{(t-1)-(K-1)} \cdot p$$

$$= \binom{t-1}{K-1} \cdot p^K (1-p)^{t-K}$$

Thus

$$P_{Y_K}(t) = \begin{cases} \binom{t-1}{K-1} p^K (1-p)^{t-K} & \text{if } t = K, K+1, \dots \\ 0 & \text{otherwise} \end{cases}$$

We call this the <u>Pascal PMF of order K</u>. <u>note</u>: range$(Y_K) = \{K, K+1, \dots\}$

<u>Exercise</u>

- sanity check $\sum_{t=K}^{\infty} P_{Y_K}(t) = 1$
- $E[Y_K] = \frac{K}{p}$
- $Var(Y_K) = \frac{K(1-p)}{p^2}$ $\Big\{$ by indep. of $T_1, \dots, T_K$
- Find CDF

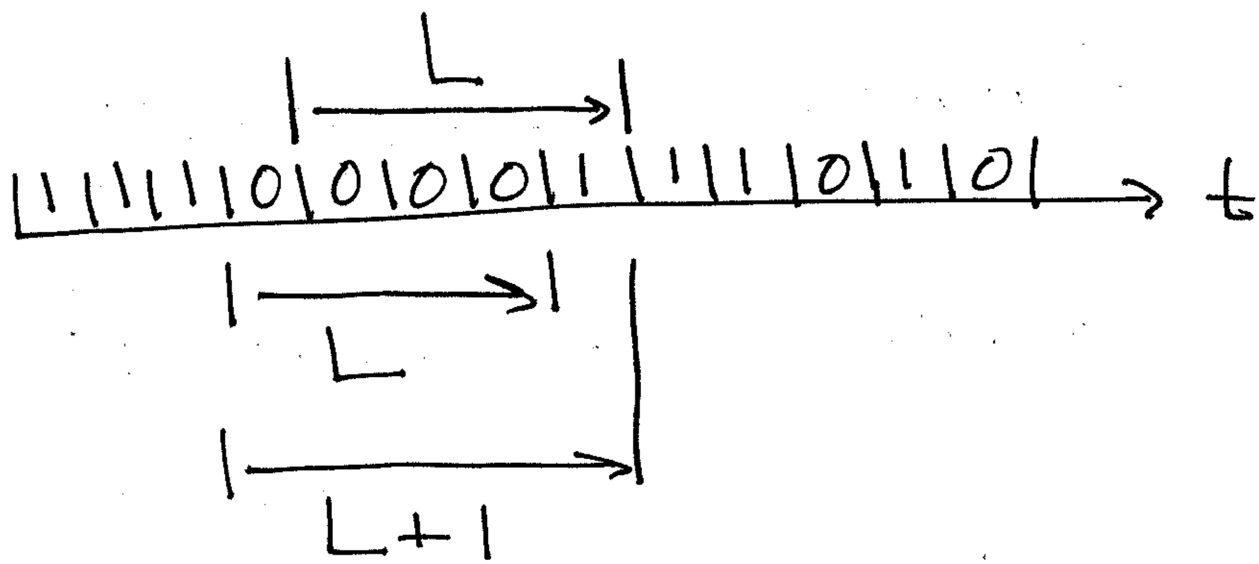
<u>note</u>: a Bernoulli Process can be defined by

- $X_1, X_2, \ldots$ Bernoulli
- $T_1, T_2, \ldots$ Geometric
- $Y_1, Y_2, \ldots$ Pascal

<u>Ex.</u>

You buy a lottery ticket every day. Let L be the length of your 1^{st} losing streak.

What is the distribution of L? i.e. find $p_L(t)$.



Is $L+1$ Geometric? No!

Why? L cannot be 0, so $L+1$ cannot be 1, so $L+1$ cannot be geometric note:

L is also # days after 1^{st} loss, up to and including next win. This is Geometric.

Splitting a Bernoulli Process

Suppose we have a Bernoulli Process (Param. p) producing a stream of arrivals of Jobs into a queue

We send each job to one of 2 servers with

$$P(\text{server } 1) = q$$

$$P(\text{server } 2) = 1 - q$$

⋮