CSE 107 5-21-26

Define

$$h(y) = Var(X \mid Y=y)$$

$$= E\left[\left(X - E[X \mid Y=y]\right)^2 \mid Y=y\right]$$

sub. Y for y, to get

$$h(Y) = Var(X \mid Y)$$ conditional variance

$$= E\left[\left(X - E[X \mid Y]\right)^2 \mid Y\right]$$

Law of total Variance

$$\boxed{Var(X) = E[Var(X|Y)] + Var(E[X|Y])}$$

Proof

using variance in terms of moments:

$$Var(X|Y) = E[X^2|Y] - E[X|Y]^2$$

$$\therefore E[X^2|Y] = Var(X|Y) + E[X|Y]^2$$

Thus

$$Var(X) = E[X^2] - E[X]^2$$

$$= E[E[X^2|Y]] - E[E[X|Y]]^2$$

$$= E[Var(X|Y) + E[X|Y]^2]$$

$$- E[E[X|Y]]^2$$

$$= E[Var(X|Y)]$$

$$+ E[E[X|Y]^2] - E[E[X|Y]]^2$$

$$= E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

$\blacksquare$