CSE 107 5-19-26

midterm 2: Thur 5-21-26

<u>Ex.</u>

Suppose we toss a coin n times, where $P(\text{head}) = p$. (indep. tosses)

Let

$$X = \#\text{heads}$$

and

$$Y = \#\text{tails}$$

compute $\rho(X, Y)$.

Solution

observe $X + Y = n$, so

$$E[X] + E[Y] = E[X+Y] = E[n] = n$$

Also

$$\begin{aligned} Var(Y) &= Var(-X+n) \\ &= (-1)^2 Var(X) \\ &= Var(X) \end{aligned}$$

Also

$$(X - E[X]) + (Y - E[Y]) = n - n = 0$$

so

$$Y - E[Y] = -(X - E[X])$$

Thus

$$\mathrm{Cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$$

$$= -E[(X-E[X])^2]$$

$$= -\mathrm{Var}(X)$$

so

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\cdot\mathrm{Var}(Y)}}$$

$$= \frac{-\mathrm{Var}(X)}{\sqrt{\mathrm{Var}(X)^2}}$$

$$= \frac{-\mathrm{Var}(X)}{\mathrm{Var}(X)} = \boxed{-1}$$

<u>Exercises</u>: Prove that

- $\mathrm{Cov}(aX+b, Y) = a\,\mathrm{Cov}(X,Y)$
- $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
- if $a>0$, then $\rho(aX+b, Y) = \rho(X,Y)$
- $\mathrm{Cov}(aX+bY, Z) = a\,\mathrm{Cov}(X,Z) + b\,\mathrm{Cov}(Y,Z)$

See handout.

Variance of a Sum

let X, Y be r.v.s (not necessarily indep.) Then

- $Var(X+Y)$
$$= Var(X) + Var(Y) + 2Cov(X,Y)$$

- if $X_1, X_2, \ldots X_n$ are r.v.s,

$$Var\left(\sum_{i=1}^{n} X_i\right)$$
$$= \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$$

note

$$\sum_{i \neq j} Cov(X_i, X_j) = 2\sum_{i<j} Cov(X_i, X_j)$$

Proof of $n=2$ case!

Let $U = X - E[X] \quad \therefore E[U]=0$
and $V = Y - E[Y] \quad \therefore E[V] = 0$

Then

$$Var(X+Y) = Var\left((U+V) + E[X+Y]\right)$$

$$= Var(U+V)$$

$$= E\left[\left((U+V) - \underbrace{E[U+V]}_{0}\right)^2\right]$$

$$= E\left[(U+V)^2\right]$$

$$= E\left[U^2 + V^2 + 2UV\right]$$

$$= E[U^2] + E[V^2] + 2E[UV]$$

$$= E\left[(X-E[X])^2\right] + E\left[(Y-E[Y])^2\right]$$

$$+ 2E\left[(X-E[X])(Y-E[Y])\right]$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

<u>Exercise</u> Prove for general n by induction.

4.3 Conditional Expectation and Variance

Recall that

$$g(y) = E[X \mid Y = y]$$

is a function. If we substitute the r.v. Y for y in $g(y)$ we get a r.v. denoted

$$g(Y) = E[X \mid Y]$$ conditional expectation

The expected value of this r.v. is

$$E[E[X|Y]] = E[g(Y)]$$

$$= \int_{-\infty}^{\infty} g(y) \cdot f_Y(y)\, dy \quad \left\{ \begin{array}{l} \text{expected} \\ \text{value rule} \end{array} \right.$$

$$= \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\, dy$$

$$= E[X]$$

Law of iterated expectations

$$\boxed{E[E[X|Y]] = E[X]}$$