CSE 107 5-15-26

Supplemental Lecture

Convolutions

Let $Z = X + Y$, X, Y indep.

Goal: find $f_Z(z)$ if X, Y cont.
and find $p_Z(z)$ if X, Y discrete.

Let X, Y be indep. discrete
r.v.s and

$$Z = X + Y$$

Then

$$p_Z(z) = P(Z = z)$$

$$= P(X + Y = z)$$

$$= \sum_{\{(x,y) \mid x+y=z\}} P(X = x, Y = y)$$

$$= \sum_{\{(x,z-x) \mid x \in \text{range}(X)\}} P(X = x, Y = z - x)$$

$$= \sum_{x} P(X = x) \cdot P(Y = z - x)$$

$$= \sum_{x} p_X(x) \cdot p_Y(z - x)$$

<u>Defn</u>

The convolution Product of

P_X, P_Y is

$$\boxed{(P_X * P_Y)(z) = \sum_x P_X(x) P_Y(z-y)}$$

note: $P_X * P_Y = P_Y * P_X$

We've shown

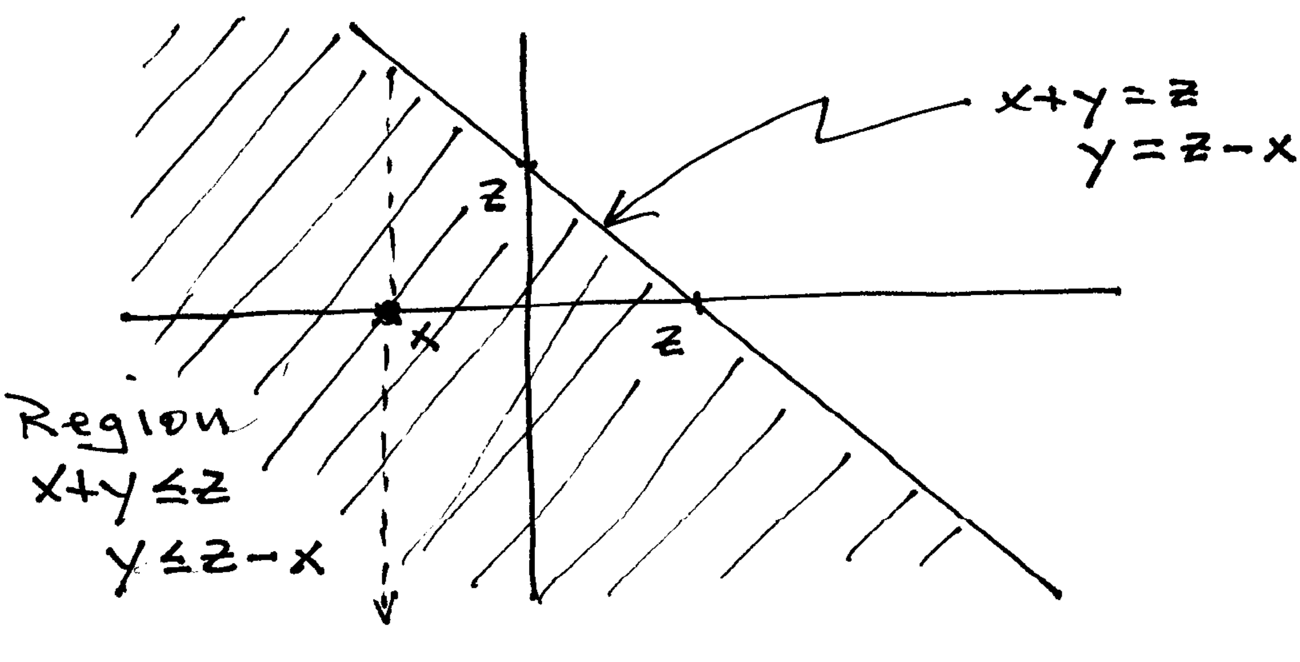
$$\boxed{P_{X+Y} = P_X * P_Y = P_Y * P_X}$$

Now let X, Y be independent, jointly continuous r.v.s, and let

$$Z = X + Y$$

Then

$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$
$$= \iint_{\{(x,y) \mid x+y \le z\}} f_{X,Y}(x,y)\,dy\,dx$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) \cdot f_Y(y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{z-x} f_Y(y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot P(Y \leq z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x)\, dx$$

Thus

$$f_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \frac{d}{dz} F_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

<u>Defn.</u>

The <u>convolution product</u> of f_X, f_Y is

$$\boxed{(f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx}$$

<u>note!</u> $f_X * f_Y = f_Y * f_X$

We've shown

$$\boxed{f_{X+Y} = f_X * f_Y = f_Y * f_X}$$

Remark

Remember, both discrete and continuous formulas for P_{X+Y} and f_{X+Y} require that X, Y be independent.

Ex.

Let X, Y be indep. continuous uniform on $[a, b]$, i.e.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and similarly for f_Y. Let

$$Z = X + Y \qquad \left\{ \begin{array}{l} \text{note:} \\ \text{range}(Z) = [2a, 2b] \end{array} \right\}$$

Find $f_Z(z)$.

Solution.

we have

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

The integ-rand is $\frac{1}{(b-a)^2}$ iff both

$$a \le x \le b \quad \text{and} \quad a \le z-x \le b$$

i.e.

$$a \le x \le b \quad \text{and} \quad z-b \le x \le z-a$$

(and integ-rand is 0 otherwise).

i.e. iff

$$\max(a, z-b) \le x \le \min(b, z-a)$$

Thus

$$f_Z(z) = \int_{\max(a,z-b)}^{\min(b,z-a)} \frac{1}{(b-a)^2}\, dx$$

$$= \frac{\min(b,z-a) - \max(a,z-b)}{(b-a)^2}$$

The interval of integration is empty if either

$$z-a < a \implies z < 2a$$

or

$$z-b > b \implies z > 2b$$

Thus for $f_Z(z)$ to be non-zero we require $2a \le z \le 2b$. Hence

$$f_Z(z) = \begin{cases} \dfrac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2} & \text{if } 2a \le z \le 2b \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: $Z = X + Y$ is not uniform.

<u>Exercise</u>

find $f_{X+Y}(z)$ where X is uniform(a,b) and Y is uniform(c,d), also X, Y indep.

Exercise

Let X, Y be discrete uniform r.v.s with $\text{range}(X) = \text{range}(Y) = \{1, 2, \ldots, m\}$. Let

$$Z = X + Y$$

Find the PMF $p_Z(n)$. [Hint: observe $\text{range}(Z) = \{2, 3, \ldots, 2m\}$]

Ex.

Let X, Y be indep. normal r.v.s with

means: μ_X, μ_Y resp.

variances: σ_X^2, σ_Y^2 resp.

Thus

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot e^{-(x-\mu_X)^2/2\sigma_X^2}$$

and similarly for Y. Let

$$Z = X + Y$$

find $f_Z(z)$.

Solution:

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y} \cdot e^{-\frac{((z-x)-\mu_y)^2}{2\sigma_y^2}}\, dx$$

= ------ (exercise, hard!!) .....

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{\sigma_x^2+\sigma_y^2}} \cdot e^{-\frac{(z-(\mu_x+\mu_y))^2}{2(\sigma_x^2+\sigma_y^2)}}$$

we conclude that $Z = X + Y$ is normal with mean $\mu_Z = \mu_X + \mu_Y$ and variance $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$.

It follows that if X, Y are independent normal r.v.s, then

$$aX + bY$$

is also normal.

4.2 Covariance and Correlation

Defn

The covariance of r.v.s X, Y is

$$\text{cov}(X,Y) = E\big[(X-E[X])(Y-E[Y])\big]$$

observe that

$$\text{cov}(X,X) = \text{Var}(X)$$

note:

- if $\text{cov}(X,Y) = 0$, we say X and Y are uncorrelated.

- if $\operatorname{cov}(X,Y) > 0$, we say X, Y are <u>positively correlated</u>.
- if $\operatorname{cov}(X,Y) < 0$, we say X, Y are <u>negatively correlated</u>.

observe

$$\operatorname{Cov}(X,Y) = E\left[(X - \underbrace{E[X]}_{\mu_X})(Y - \underbrace{E[Y]}_{\mu_Y})\right]$$

$$= E\left[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y\right]$$

$$= E[XY] - \mu_X \underbrace{E[Y]}_{\mu_Y} - \mu_Y \underbrace{E[X]}_{\mu_X} + \mu_X \mu_Y$$

$$= E[XY] - \mu_X \mu_Y = E[XY] - E[X] \cdot E[Y]$$

Thus

$$\boxed{Cov(X,Y) = E[XY] - E[X]E[Y]}$$

<u>note</u>

X, Y independent $\Rightarrow E[XY] = E[X] \cdot E[Y]$

$\Rightarrow Cov(X,Y) = 0$

$\Rightarrow X, Y$ are uncorrelated.

The converse is false however.

See Ex. 4.13 on P. 218-219.

Contrapositive is true:

$\boxed{\text{X, Y Pos. or neg. correlated}} \Rightarrow \boxed{\text{X, Y not independent}}$

Defn

The <u>correlation coefficient</u> $\rho(X,Y)$ is

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

<u>note</u> $\rho(X,Y)$ is a normalized version of covariance.

Theorem

Let X, Y be r.v.s, then

$$-1 \le \rho(X,Y) \le 1$$

-1: high neg. correlation.

1: high pos. correlation