CSE 107 5-14-26

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midterm2 : Thur. 5/21

continuing example ....

(2) diff. w.r.t. t to get

$$f_T(t) = -f_R\left(\frac{180}{t}\right)\cdot\left(-\frac{180}{t^2}\right)$$

We are given

$$f_R(r) = \begin{cases} \frac{1}{30} & 30 \le r \le 60 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$f_T(t) = \begin{cases} -\frac{1}{30} \cdot \left(-\frac{180}{t^2}\right) & 30 \le \frac{180}{t} \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6}{t^2} & \frac{1}{60} \le \frac{t}{180} \le \frac{1}{30} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6}{t^2} & \text{if } 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Lemma

If $Y = aX + b$, then

$$\boxed{f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)} \quad (\text{if } a \neq 0)$$

Proof

① $F_Y(y) = P(Y \leq y)$

$= P(aX + b \leq y)$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) & a > 0 \\ P\left(X \geq \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a>0 \\ 1 - P\left(X < \frac{y-b}{a}\right) & a<0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a>0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a<0 \end{cases}$$

② diff w.r.t. y :

$$f_Y(y) = \begin{cases} f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} & (a>0) \\ -f_X\left(\frac{y-b}{a}\right) \frac{1}{a} & (a<0) \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Ex.

Let X be normal with mean μ, and std. dev. σ. Then

$$Y = aX + b$$

is normal with mean $a\mu + b$ and std. dev. $|a| \cdot \sigma$ (i.e. Var $a^2\sigma^2$).

Proof

We know

$$f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

so

$$f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}}$$

$= \ldots$ exercise $\ldots$

$$= \frac{1}{\sqrt{2\pi} \cdot |a|\sigma} e^{-\frac{(y-(a\mu+b))^2}{2(a\sigma)^2}}$$ which is normal.

$\blacksquare$

<u>Defn</u>

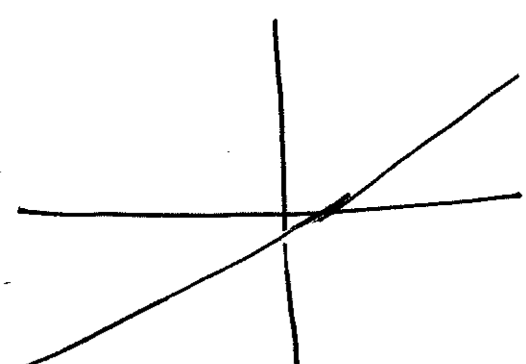
A function $g: \mathbb{R} \to \mathbb{R}$ is called <u>strictly monotic</u> iff for all $x_1, x_2 \in \mathbb{R}$, either

(1) $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$ (increasing)

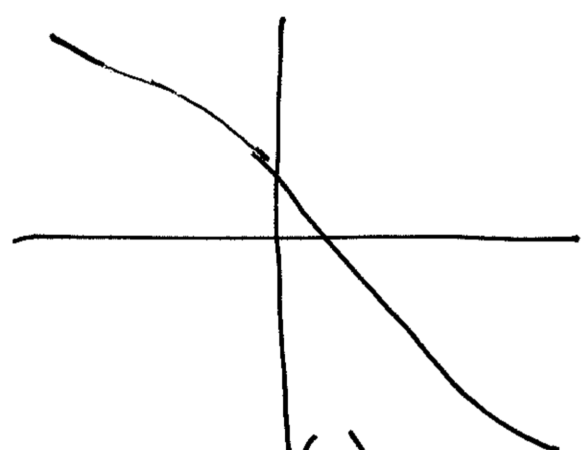
or

(2) $x_1 < x_2 \Rightarrow g(x_1) > g(x_2)$ (decreasing)

Such a function is necessarily bijective, hence invertible.



(1)



(2)

Let $h = g^{-1}$ i.e.

$$h(g(x)) = x \quad \text{for all } x \in \mathbb{R}$$

and

$$g(h(y)) = y \quad \text{for all } y \in \mathbb{R}$$

Lemma

If $g: \mathbb{R} \to \mathbb{R}$ is strictly motonic,

then

(1) $\Rightarrow$ $h = g^{-1}$ is also increasing

(2) $\Rightarrow$ $h = g^{-1}$ is also decreasing.

Proof (exercise)

Lemma

if g is monotonic and differentiable then

$$(1) \Leftrightarrow g' > 0 \Leftrightarrow h' > 0$$

and

$$(2) \Leftrightarrow g' < 0 \Leftrightarrow h' < 0$$

Proof (exercise)

Theorem

Let X be a cont. r.v., $g: \mathbb{R} \to \mathbb{R}$ strictly monotonic, and $Y = g(X)$. Then

$$\boxed{f_Y(y) = f_X(h(y)) \cdot |h'(y)|}$$

Proof

Suppose $g : \mathbb{R} \to \mathbb{R}$ is increasing i.e. case (1). Then

① $F_Y(y) = P(Y \le y)$

$$= P(g(X) \le y)$$

$$= P\left(g^{-1}(g(X)) \le g^{-1}(y)\right)$$

$$= P(X \le h(y))$$

$$= F_X(h(y))$$

② diff. w.r.t. y :

$$f_Y(y) = f_X(h(y)) \cdot h'(y) = f_X(h(y)) \, |h'(y)|$$

Since g inc $\Rightarrow$ h inc. $\Rightarrow$ $h' > 0$.

<u>exercise</u>: complete the proof in the case (2) where g is strictly decreasing

$\blacksquare$

<u>note</u>

All of this can be re-done with $g: I \to J$ where I, J are intervals.

Ex.

suppose $Y = X^2$ where X is a continuous r.v. with range$(X) \subseteq [0,\infty)$. Then also range$(Y) \subseteq [0,\infty)$, and $g(x)=x^2$ is strictly increasing on $[0,\infty)$. Thus g is invertible with inverse $h(y)=\sqrt{y}$. Thus

$$\boxed{f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}} \quad (y \geq 0)$$

Exercise

Suppose $Y = X^3$, and find $f_Y(y)$ in terms of $f_X(\cdot)$

Two random variables: $Z = g(X, Y)$.

Assume X, Y are jointly continuous r.v. s.

Ex.

Let X, Y be independent r.v.s, both uniform on $[0,1]$.

Thus

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

and similarly for Y.

Define Z by

$$Z = \max(X, Y)$$

observe $\text{range}(Z) = [0, 1]$. so

if $0 \le z \le 1$

$$\begin{aligned}
F_Z(z) &= P(Z \le z) \\
&= P(\max(X,Y) \le z) \\
&= P(X \le z,\ Y \le z) \\
&= P(X \le z) \cdot P(Y \le z) \quad \left\{ \begin{array}{l} \text{by} \\ \text{indep.} \\ \text{of } X, Y \end{array} \right. \\
&= F_X(z) \cdot F_Y(z) \\
&= z^2
\end{aligned}$$

hence

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ z^2 & \text{if } 0 \leq z < 1 \\ 1 & \text{if } z \geq 1 \end{cases}$$

diff. w.r.t. z to get

$$f_Z(z) = \begin{cases} 2z & \text{if } 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise

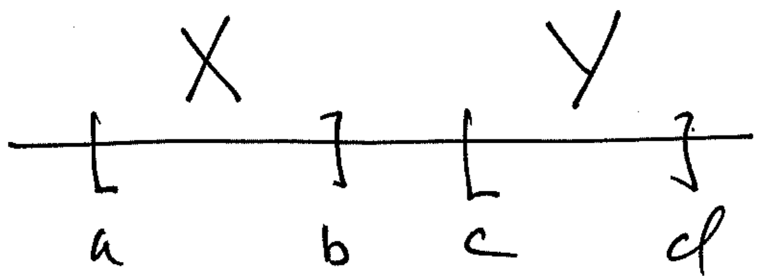
Let X be Uniform on $[a, b]$, and Y be Uniform on $[c, d]$.

Find the PDF of

$$Z = \max(X, Y)$$

assuming X, Y independent.

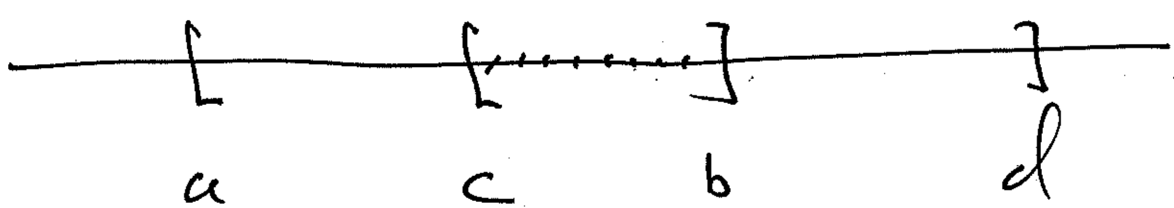
<u>Note</u>: when $b<c$, the intervals $[a,b]$ and $[c,d]$ are disjoint



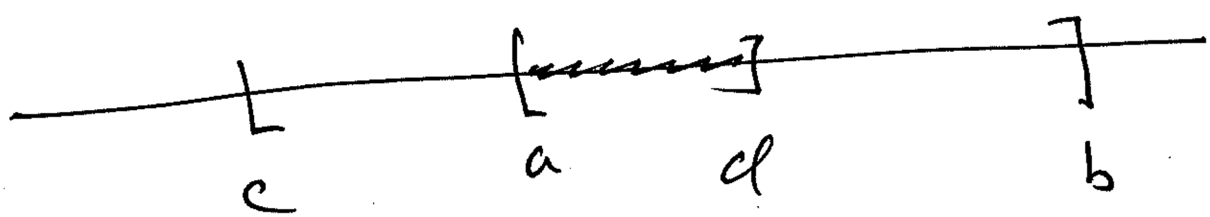
In this case $Z = \max(X,Y) = Y$, so $f_Z = f_Y$. Similarly, if $d<a$ then $f_Z = f_X$

So assume both $\boxed{c \le b}$ and $\boxed{a \le d}$, and hence

$$[a,b] \cap [c,d] \neq \phi .$$



or



<u>note</u>

range(Z) = ... ? ...