CSE 107 5-12-26

Independence

Let X, Y be jointly continuous r.v.s.

Defn

We say X, Y are independent iff

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

for all $x \in range(X)$, $y \in range(Y)$

Equivalently

$$f_{X|Y}(x|y) = f_X(x) \quad \begin{cases} \text{requires} \\ f_Y(y) > 0 \end{cases}$$

or

$$f_{Y|X}(y|x) = f_Y(y) \quad \begin{cases} \text{given} \\ f_X(x) > 0 \end{cases}$$

Can generalize to 3, 4, ..., n r.v.s.

If X, Y are independent, then any events definable in terms of X and Y are independent.

For instance

$$\{X \in I\} \text{ and } \{Y \in J\}$$

are independent for $I \subseteq \mathbb{R}$ $J \subseteq \mathbb{R}$. In particular if

$$I = (-\infty, x] \text{ and } J = (-\infty, y] :$$

then

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\, ds\, dt$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_X(s)\cdot f_Y(t)\,ds\,dt \quad \left\{\begin{array}{l}\text{by} \\ \text{indep.} \\ \text{of } X, \\ Y\end{array}\right.$$

$$= \int_{-\infty}^{y}\left(\int_{-\infty}^{x} f_X(s)\,ds\right) f_Y(t)\,dt$$

$$= \left(\int_{-\infty}^{x} f_X(s)\,ds\right)\cdot\left(\int_{-\infty}^{y} f_Y(t)\,dt\right)$$

$$= P(X \le x)\cdot P(Y \le y)$$

$$= F_X(x)\cdot F_Y(y)$$

<u>note</u>

converse is also true, i.e. if the joint CDF factors then the joint PDF factors.

Consequences of indep. of X, Y:

(1) $E[XY] = E[X] \cdot E[Y]$

(2) $E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$

(3) $Var(X+Y) = Var(X) + Var(Y)$

Exercises

- Prove (1) in continuous case.

- (hard!) Prove X, Y indep. implies $g(X), h(Y)$ indep. which gives (2). [Hint: adapt proof in Prob. #44, ch 2, p. 134 to cont. case.]

- Prove (3) in cont. case.

3.6 Bayes Rule

Recall given $A, B \subseteq \Omega$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

If $A_1, \ldots, A_n$ form a Partition of Ω then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

Thus for any $k=1,\dots,n$:

$$\boxed{P(A_k|B) = \frac{P(B|A_k)\cdot P(A_k)}{\sum_{i=1}^{n} P(B|A_i)\cdot P(A_i)}}$$

If X, Y are discrete r.v.s, then this becomes

$$\boxed{p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\, p_X(x)}{\sum_t p_{Y|X}(y|t)\cdot p_X(t)}}$$

If X, Y are continuous r.v.s, then

$$\boxed{f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) \cdot f_X(t)\, dt}}$$

<u>Ex.</u>

Let Y be Normal with std. dev. 1, and mean X, another r.v.. Suppose X is cont. uniform on $[1, 3]$.

a) find $f_Y(y)$

<u>solution</u>

We are given

$$f_X(x) = \begin{cases} \frac{1}{2} & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2} & \begin{cases} \text{if} \\ 1 \le x \le 3 \\ y \in \mathbb{R} \end{cases} \\ 0 & \begin{cases} \text{otherwise} \\ y \in \mathbb{R} \end{cases} \end{cases}$$

Thus

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\, dx$$

$$= \frac{1}{2} \int_1^3 \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}}\, dx$$

$$= \frac{1}{2} \int_1^3 \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}}\, dx \quad \left\{ \begin{array}{l} u = x-y \\ du = dx \\ x=1 \Rightarrow u = 1-y \\ x=3 \Rightarrow u = 3-y \end{array} \right.$$

$$= \frac{1}{2} \int_{1-y}^{3-y} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$$

$$= \boxed{\frac{1}{2}\left(\Phi(3-y) - \Phi(1-y)\right)}$$

b.) find $f_{X|Y}(x|y)$

Solution

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{f_Y(y)}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2} \cdot \cancel{\frac{1}{2}}}{\cancel{\frac{1}{2}} \cdot \left(\Phi(3-y) - \Phi(1-y)\right)}$$

$$= \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)} & \text{if } 1 \le x \le 3,\ y \in \mathbb{R} \\ 0 & \text{otherwise},\ y \in \mathbb{R} \end{cases}$$

c.) Suppose we sample Y and get $y=3$. What is the probability that $X \le 2$.

ans.: $\boxed{0.2848}$

d.) find $E[Y]$

ans: $\boxed{2}$

4.1 Derived Distributions

If $Y = g(X)$ how can we compute the PDF of Y

Want: $f_Y(y)$

from PDF of X : $f_X(x)$

We call $f_Y(y)$ a derived distribution

2-step Process

1.) find the CDF $F_Y(y)$

$$F_Y(y) = P\left(g(X) \le y\right)$$

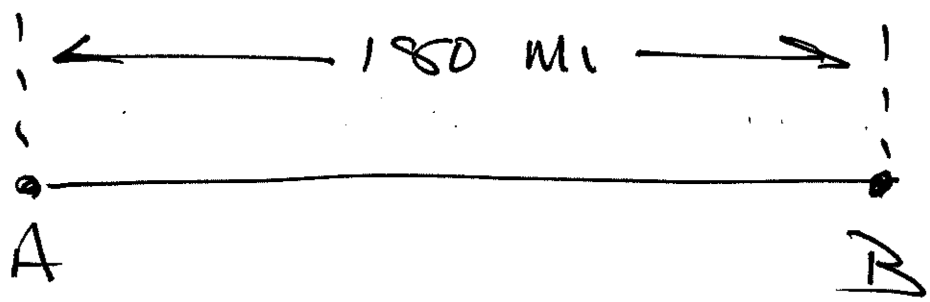
$$= \int_{\{x \mid g(x) \le y\}} f_X(x)\,dx$$

2.) differentiate w.r.t. y

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Ex.

A car travels 180 miles at a const. rate R. R is itself a r.v. which is uniform on $30 \le r \le 60$ (mph)



Let T be the time of travel. Find $f_T(t)$.

<u>Solution</u>

from $180 = R \cdot T$ we get

$$T = \frac{180}{R}$$

(1) $F_T(t) = P(T \leq t)$

$$= P\left(\frac{180}{R} \leq t\right)$$

$$= P\left(R \geq \frac{180}{t}\right)$$

$$= 1 - P\left(R < \frac{180}{t}\right)$$

$$= 1 - F_R\left(\frac{180}{t}\right)$$

(2) diff. w.r.t. t

$$f_T(t) = -f_R\left(\frac{180}{t}\right)\cdot\left(-\frac{180}{t^2}\right)$$

$\vdots$