CSE 107 5-1-26

Supplemental Lecture

<u>Ex.</u>

You take a test 3 times and your final score is the maximum of the 3 scores.

$$X = \max(X_1, X_2, X_3)$$

where X_i is score on test i. Let $\underline{\text{range}}(X_i) = \{1, 2, 3, \dots, 10\}$. Suppose each score is uniform over $\{1, \dots, 10\}$, also individual scores are independent.

Find PMF $P_X(k)$ of X.

Solution

note

$$
P_{X_i}(k) = \begin{cases} \frac{1}{10} & k = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}
$$

hence

$$
F_{X_i}(k) = \frac{k}{10} \quad (\text{for } k = 1, 2, .., 10)
$$

i.e.

$$
F_{X_i}(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{\lfloor x \rfloor}{10} & \text{if } 1 \le x \le 10 \\ 1 & \text{if } x > 10 \end{cases}
$$

<u>Plan</u>: 1st find CDF $F_X(k)$, then difference it to get $P_X(k)$

$$P_X(k) = F_X(k) - F_X(k-1)$$

so

$$F_X(k) = P(X \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k) \quad \text{by indep.}$$

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \left(\frac{k}{10}\right)^3$$

So

$$P_X(k) = \begin{cases} \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3 & \text{if } k = 1, 2, .., 10 \\ 0 & \text{otherwise} \end{cases}$$

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<u>Goal</u>: Compare Exponential(λ) to Geometric(p).

Recall if X is Geometric(p), then

$$P_X(k) = p(1-p)^{k-1} \quad (k = 1, 2, 3, \ldots)$$

and

$$F_X(n) = \sum_{k=1}^{n} p(1-p)^{k-1}$$

$$= p \cdot \frac{1-(1-p)^n}{1-(1-p)}$$

$$= 1-(1-p)^n$$

If Y is Exponential (λ), then

$$f_Y(x) = \lambda e^{-\lambda x} \quad (\text{if } x \geq 0)$$

Thus

$$F_Y(x) = \int_{-\infty}^{x} \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t}\Big|_0^x$$

$$= -\left(e^{-\lambda x} - 1\right)$$

$$= 1 - e^{-\lambda x} \quad (\text{if } x \geq 0)$$

To compare F_X to F_Y, let $\delta > 0$ be a positive real number satisfying

$$\boxed{e^{-\lambda\delta} = 1-p} \quad (\text{i.e. } p = 1-e^{-\lambda\delta})$$

$$\therefore \quad -\lambda\delta = \ln(1-p)$$

$$\therefore \quad \delta = \frac{-\ln(1-p)}{\lambda}$$

For this value of δ we have

$$(e^{-\lambda\delta})^n = (1-p)^n$$

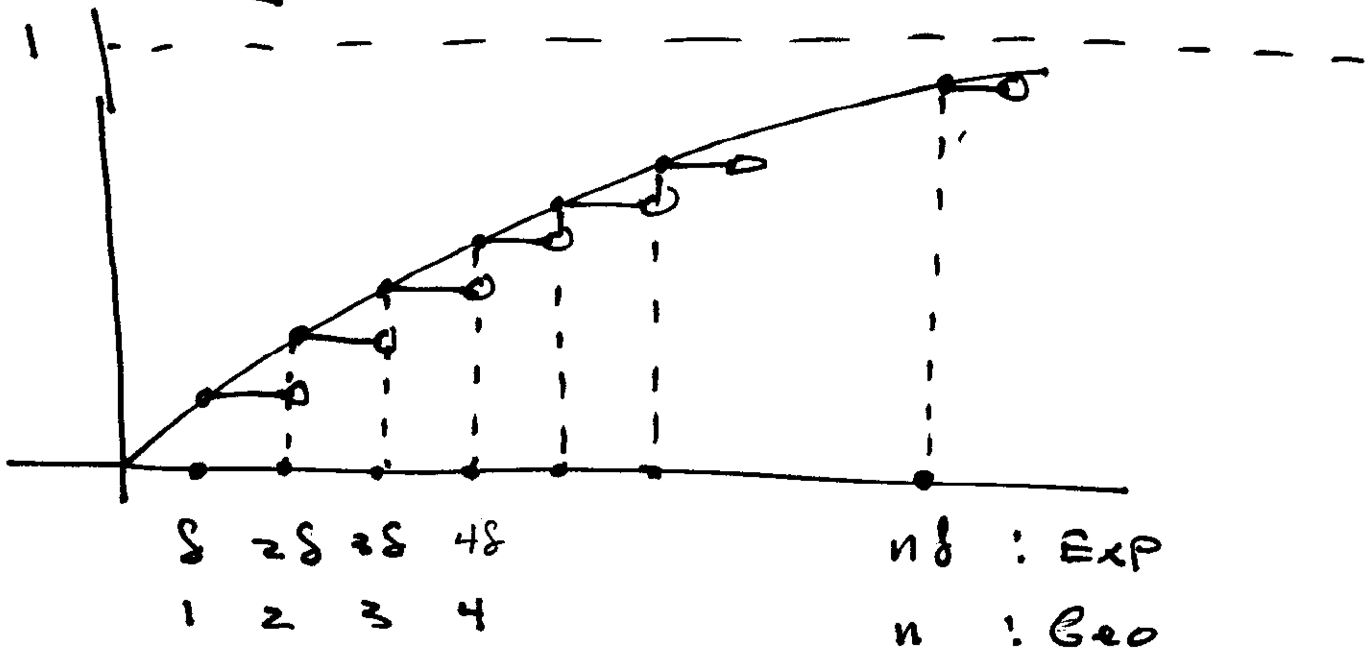
$$1-e^{-\lambda\delta n} = 1-(1-p)^n$$

Thus

$$F_Y(\delta n) = F_X(n)$$

takes place of x

Graphing both



3.3 Normal Random Variables

Let X be a r.v. with PDF

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}}$$

We call X Normal (or Gaussian) with Parameters μ, σ.

It can be shown that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

One can also show

$$E[X] = \mu$$

and

$$Var(X) = \sigma^2$$

$$\therefore \quad \sqrt{Var(X)} = \sigma$$

A normal random variable is called <u>standard Normal</u> iff $\mu = 0, \ \sigma = 1$

Theorem

If X is Normal(μ, σ), and if $a \neq 0$, and b are constants, then

$$Y = aX + b$$

is Normal with $E[Y] = a\mu + b$ and Variance $Var(Y) = a^2\sigma^2$, i.e. Y is Normal$(a\mu + b, |a|\sigma)$.

Proof

assume $a > 0$. (exercise: $a < 0$).

Then

$$F_Y(y) = P(Y \le y)$$
$$= P(aX + b \le y)$$
$$= P\left(X \le \frac{y-b}{a}\right)$$

$$\therefore \quad F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

$$\Big\downarrow \frac{d}{dy}$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{\frac{-\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi} \cdot (a\sigma)} \cdot e^{\frac{-\left(\frac{y-b-a\mu}{a}\right)^2}{2\sigma^2}}$$

$$= \boxed{\frac{1}{\sqrt{2\pi}(a\sigma)} \cdot e^{\frac{-(y-(a\mu+b))^2}{(a\sigma)^2}}}$$

Thus Y is $\text{Normal}(a\mu+b, a\sigma)$

$\blacksquare$

Exercise: do case $a<0$.