CSE 107 4-7-26

1.5 Independence

If knowledge of $B \subseteq \Omega$ tells us nothing about $A \subseteq \Omega$, we say A, B are independent.

$$(*) \qquad P(A|B) = P(A)$$

i.e.

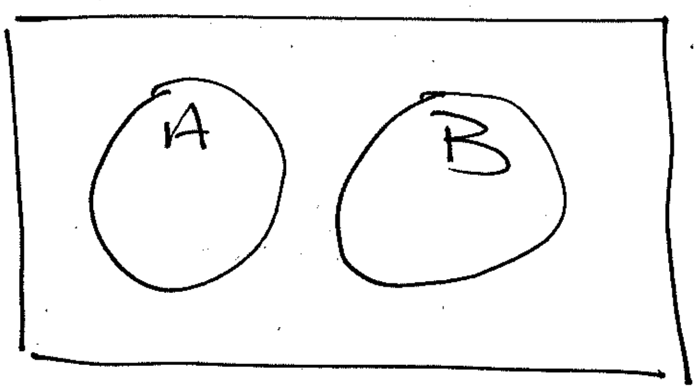
$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$(**) \qquad \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Defn

We say $A, B \subseteq \Omega$ are **independent** iff ** holds.

note: $A \cap B = \phi \not\Rightarrow A, B$ indep.

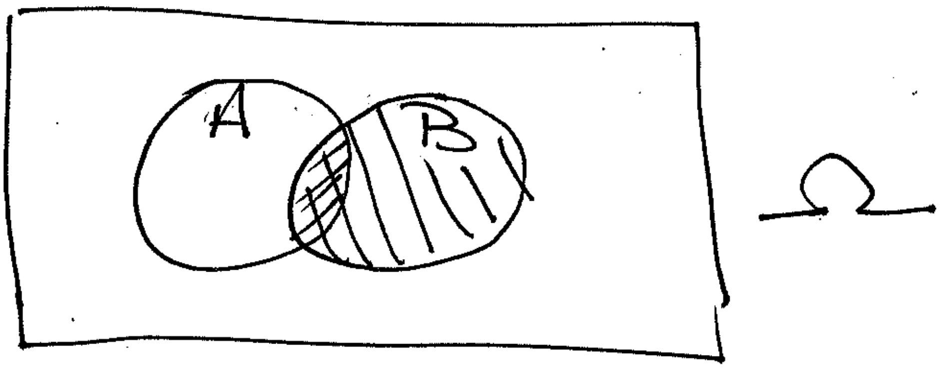


$A \cap B = \phi$

$P(A) \neq 0$
$P(B) \neq 0$
$P(A \cap B) = 0$

Thus $P(A \cap B) \neq P(A) \cdot P(B)$

<u>Picture</u>: $P(\cdot) \approx \text{area}(\cdot)$



$$P(A|B) = P(A)$$

$$\frac{\text{area}(A\cap B)/\text{area}(\Omega)}{\text{area}(B)/\text{area}(\Omega)} = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

i.e. $$\frac{\text{area}(A\cap B)}{\text{area}(B)} = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

Ex. Throw two fair dice.

Let

$$A_i = \{1^{st} \text{ die is } i\} \quad 1 \le i \le 6$$

$$B_j = \{2^{nd} \text{ die is } j\} \quad 1 \le j \le 6$$

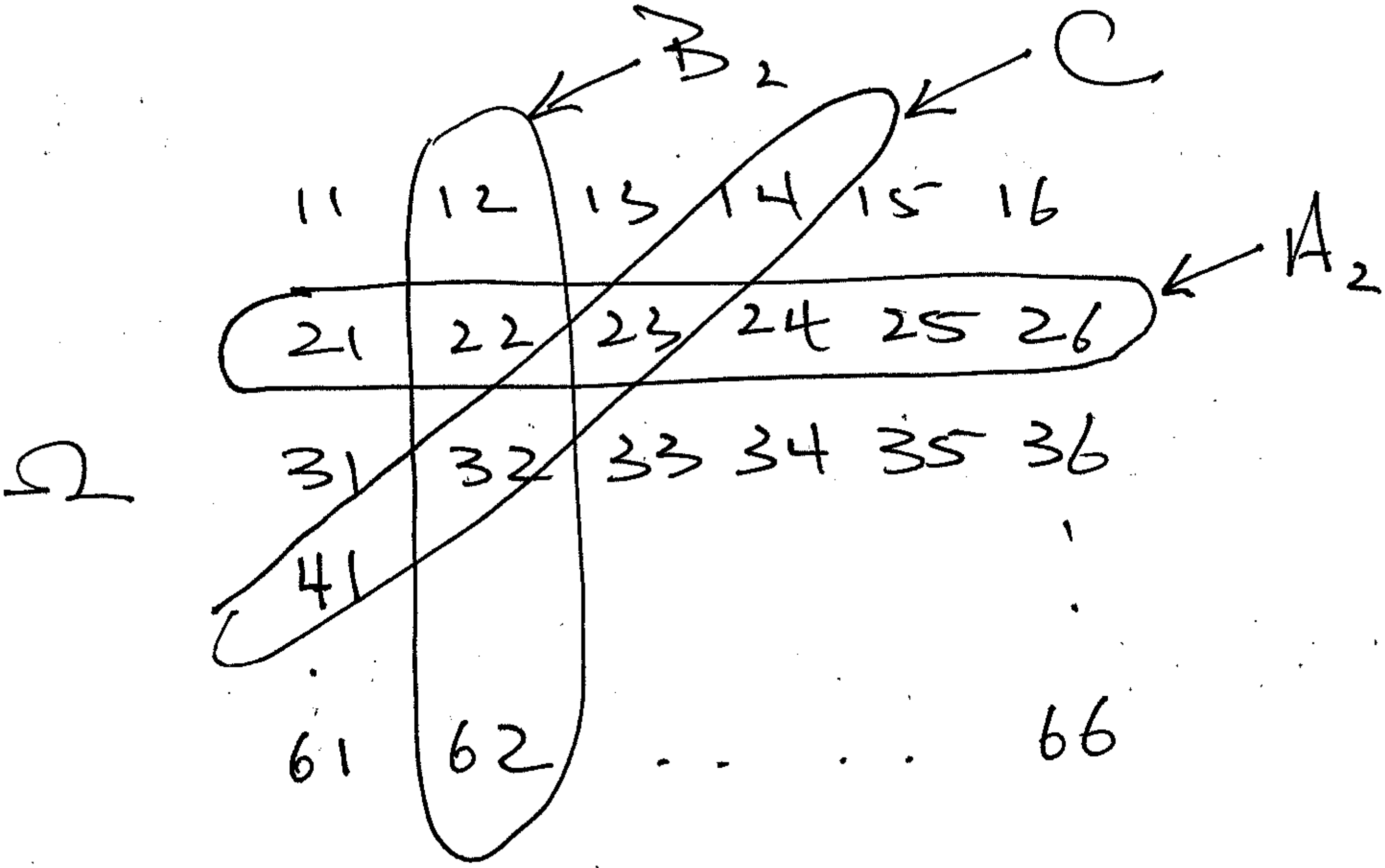
$$P(A_i) = \frac{6}{36} = \frac{1}{6}$$

$$P(B_j) = \frac{6}{36} = \frac{1}{6}$$

If A_i, B_j are independent, then

$$P((i,j)) = P(A_i \cap B_j) = P(A_i) \cdot P(B_j)$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$



Thus Independence $\Longrightarrow$ discrete unif. Prob. law and conversely ($\Longleftarrow$)

Let $C = \{sum = 5\}$

$= \{14, 23, 32, 41\}$

$\therefore P(C) = \frac{4}{36} = \frac{1}{9}.$

- are C, A_2 independent?

$A_2 = \{21, 22, 23, 24, 25, 26\}$

$A_2 \cap C = \{23\}$

$\therefore P(A_2 \cap C) = \frac{1}{36}$

$P(A_2) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54} \neq \frac{1}{36}$

answer: $\boxed{\text{no}}$

Let $D = \{\text{doubles}\}$

$= \{11, 22, 33, 44, 55, 66\}$

- Are D, A_2 independent?

$$P(D) = \frac{6}{36} = \frac{1}{6}$$

$$D \cap A_2 = \{22\}$$

$$\therefore P(D \cap A_2) = \frac{1}{36}$$

$$P(D) \cdot P(A_2) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

answer: $\boxed{\text{yes}}$

Exercise

Let $A, B \subseteq \Omega$. Prove the following are equivalent (T.F.A.E)

(a) A, B are independent.

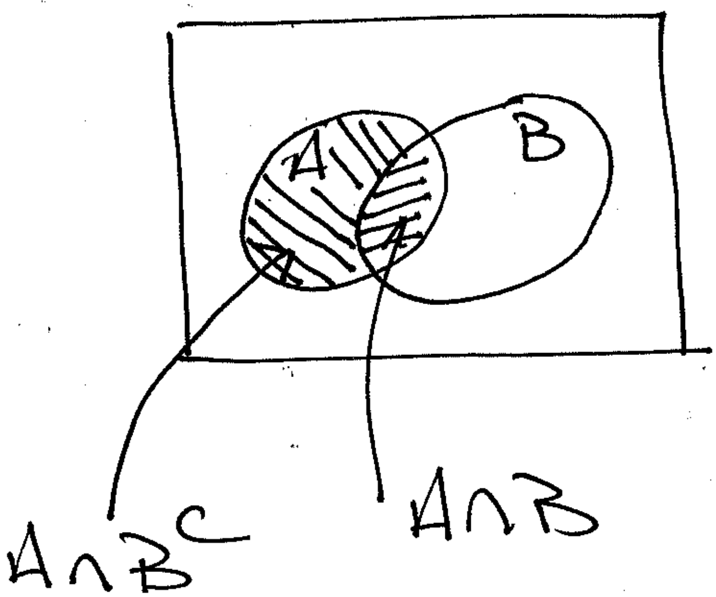
(b) A, B^c " " .

(c) A^c, B " " .

(d) A^c, B^c " " .

Proof that (a) $\Rightarrow$ (b)

note $A = (A \cap B) \cup (A \cap B^c)$ is a disjoint union, i.e. $(A \cap B) \cap (A \cap B^c) = \phi$



By axiom (iii):

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$= \underbrace{P(A) \cdot P(B)}_{\text{by (a)}} + P(A \cap B^c)$$

$$\therefore P(A \cap B^c) = P(A) - P(A) \cdot P(B)$$

$$= P(A) \cdot (1 - P(B))$$

$$= P(A) \cdot P(B^c)$$

which gives (b). ∎

Conditional Independence

Recall, if $P(C) > 0$, then

$$P(\cdot \mid C)$$

is a valid probability law.

Defn

We say $A, B \subseteq \Omega$ are <u>conditionally independent given C</u> iff

$$\boxed{P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)}$$

Warning

$$\boxed{\begin{array}{l}\text{cond.}\\ \text{indep.}\\ \text{of } A,B\\ \text{given } C\end{array}} \;\; \begin{array}{c}\not\Longrightarrow\\[1em] \not\Longleftarrow\end{array} \;\; \boxed{\begin{array}{l}\text{indep. of}\\ A, B\end{array}}$$

See examples 1.20, 1.21 in text, p. 37.

Independence of multiple events

What does it mean for 3 events $A, B, C \subseteq \Omega$ to be independent.

Defn

We say A, B, C are <u>independent</u> iff the following are true

$$(1)\ P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$(2)\ P(A \cap B) = P(A) \cdot P(B)$$

$$(3)\ P(A \cap C) = P(A) \cdot P(C)$$

$$(4)\ P(B \cap C) = P(B) \cdot P(C)$$

<u>Pairwise indep. of A, B, C</u>

Warning

$$\boxed{\begin{array}{l}\text{Pairwise} \\ \text{indep. of} \\ A, B, C \\ \text{i.e. (2), (3), (4)}\end{array}} \;\; \begin{array}{c} \not\Longrightarrow \\ \not\Longleftarrow \end{array} \;\; \boxed{\begin{array}{l}\text{equation (1)} \\ P(A \cap B \cap C) = \\ P(A) \cdot P(B) \cdot P(C)\end{array}}$$

see examples 1.22, 1.23

in text, p. 39

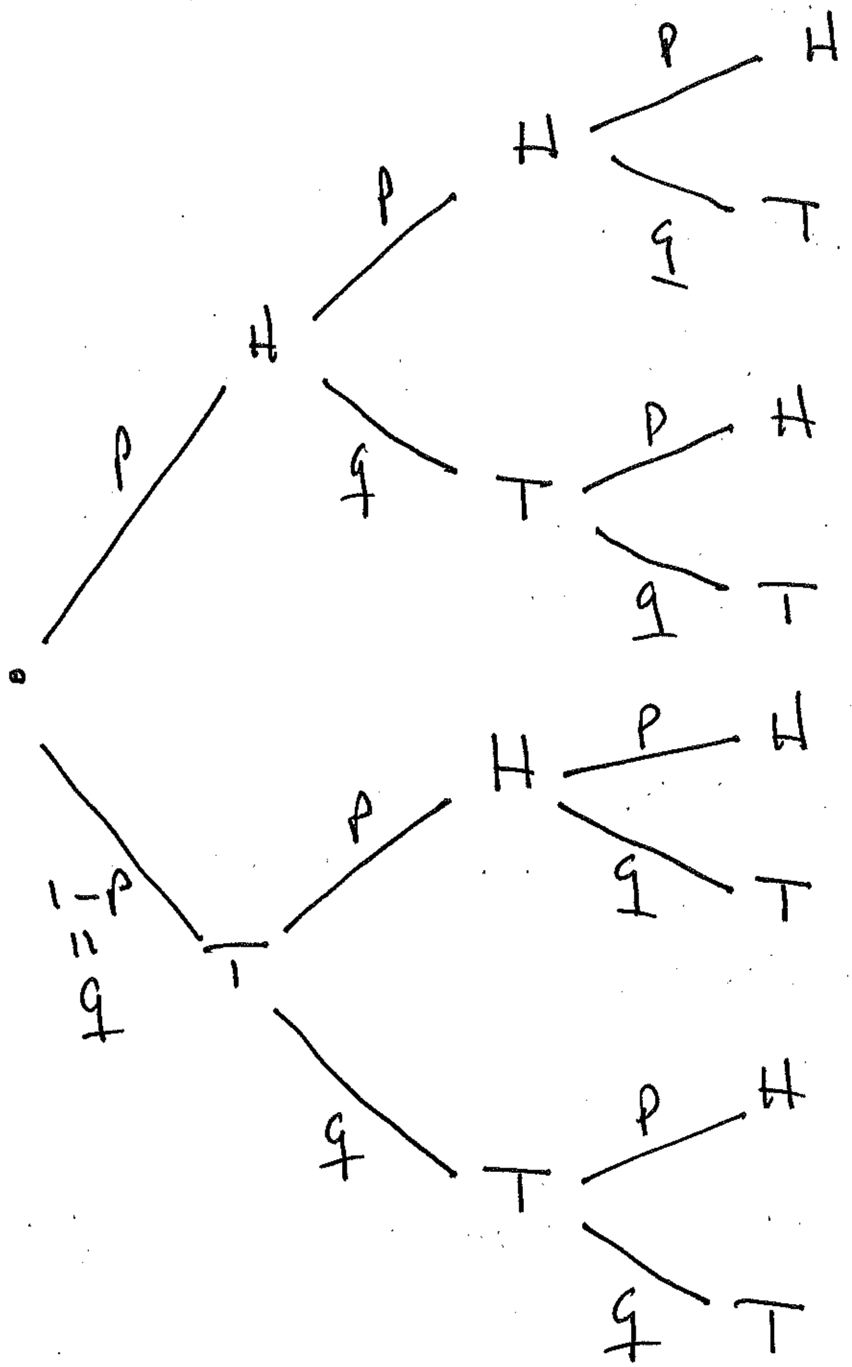
Binomial Probabilities

A simple experiment with exactly 2 outcomes is called a Bernoulli Trial.

$$\text{outcomes} = \begin{cases} \text{success} = 1 = \text{heads} \\ \text{failure} = 0 = \text{tails} \end{cases}$$

Ex. Perform 3 independent tosses of a weighted coin, with

$$P(\text{head}) = p$$
$$P(\text{tail}) = 1-p = q$$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