CSE 107 4-30-26

Ex Alice's driving time

$$f_X(x) = \begin{cases} \frac{1}{6} & \text{if } 40 \le x \le 44 \\ \frac{1}{18} & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{40}^{44} x \cdot \frac{1}{6}\,dx + \int_{44}^{50} x \cdot \frac{1}{18}\,dx$$

$$= \frac{1}{6} \cdot \frac{1}{2} x^2 \Big|_{40}^{44} + \frac{1}{18} \cdot \frac{1}{2} x^2 \Big|_{44}^{50}$$

$$= \frac{1}{12}\left(44^2 - 40^2\right) + \frac{1}{36} \cdot \left(50^2 - 44^2\right) = \cdots = \boxed{\frac{131}{3}}$$

Also

$$E[X^2] = \frac{17{,}228}{9} \quad \text{(exercise)}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{67}{9} \quad \text{(exercise)}$$

<u>Ex</u> Exponential r.v.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

here $\lambda > 0$ is a parameter.

<u>note</u>: Exponential is continuous time analog of Geometric.

<u>Note</u>:

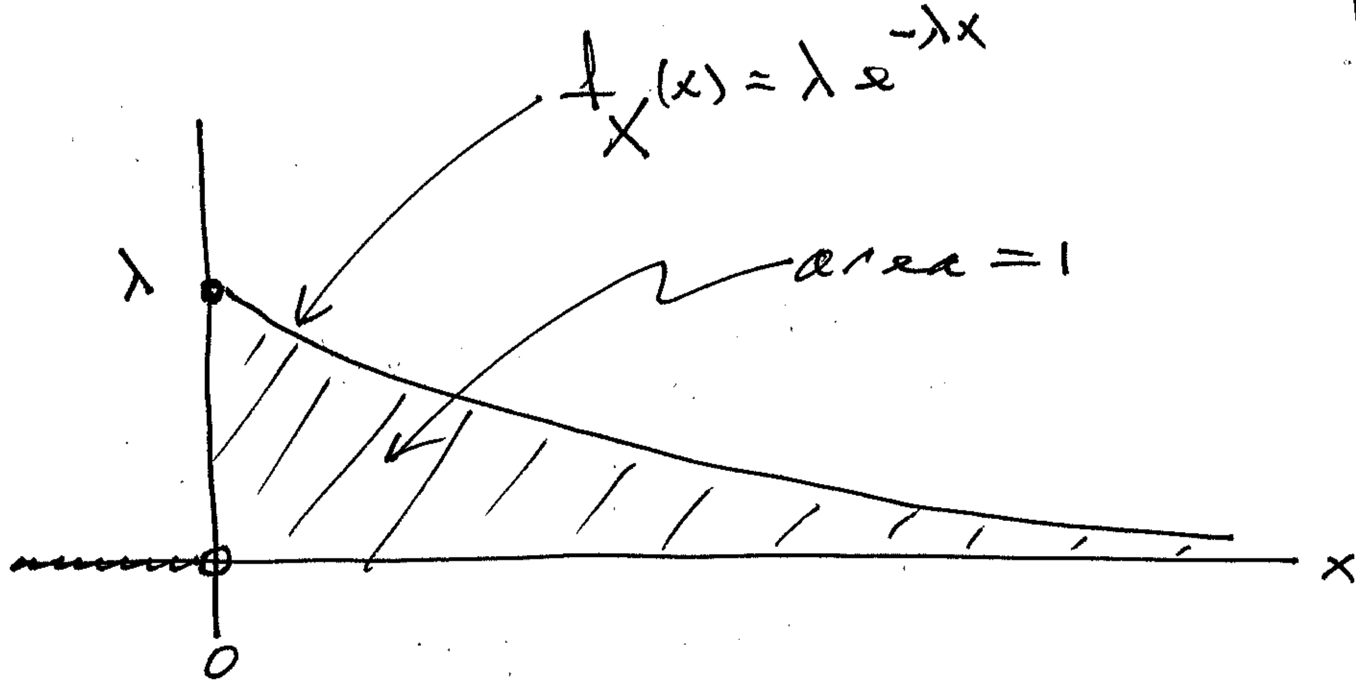
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left(-\frac{1}{\lambda}\right)\cdot e^{-\lambda x}\Big|_0^{\infty}$$

$$= -\lim_{a\to\infty} e^{-\lambda x}\Big|_0^{a}$$

$$= -\lim_{a\to\infty}\left(e^{-\lambda a} - 1\right)$$

$$= -(0-1) = 1 \quad \checkmark$$



<u>Lemma</u>

- $E[X] = \frac{1}{\lambda}$ ✓
- $E[X^2] = \frac{2}{\lambda^2}$ (exercise)
- $Var(X) = \frac{1}{\lambda^2}$ (exercise)

integration by parts.

Proof of $E[X] = \frac{1}{\lambda}$:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx$$

$$= -\int_0^{\infty} x(-\lambda e^{-\lambda x})\,dx$$

$$\boxed{\begin{array}{ll} u = x & dv = -\lambda e^{-\lambda x}dx \\ du = dx & v = e^{-\lambda x} \end{array}}$$

$$= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= -(0-0) + \left(-\frac{1}{\lambda}\right)e^{-\lambda x}\Big|_0^{\infty}$$

$$= \frac{-1}{\lambda}(0-1) = \frac{1}{\lambda}$$ ▨

<u>Note</u>

- X has units $\dfrac{\text{time}}{\text{arrival}}$
- $E[X]$ " " $\dfrac{\text{time}}{\text{arrival}}$
- $\dfrac{1}{\lambda}$ " " $\dfrac{\text{time}}{\text{arrival}}$
- λ " " $\dfrac{\text{arrivals}}{\text{time}}$

Thus λ is the <u>arrival rate</u>.

Lemma

If X is Exponential(λ), then

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \checkmark \\ 1 & \text{if } a \leq 0 \checkmark \end{cases}$$

Proof

Let $a > 0$. Then

$$P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x}\, dx$$

$$= \lambda \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_a^\infty$$

$$= -\left(0 - e^{-\lambda a}\right) = e^{-\lambda a}.$$

$\blacksquare$

Ex.

The time X until the next earthquake (4.0 or above, world-wide) is modeled as an exponential r.v. with a mean of 53 minutes.

What is the probability of such an earthquake within next 10 minutes?

Solution

$$E[X] = \frac{1}{\lambda} = 53 \Rightarrow \lambda = \frac{1}{53}$$

hence

$$P(0 \le X \le 10) = \int_0^{10} \lambda e^{-\lambda x} dx$$

$$= \int_0^{\infty} (\quad) dx - \int_{10}^{\infty} (\quad) dx$$

$$= P(X \ge 0) - P(X \ge 10)$$

$$= 1 - e^{-\lambda \cdot 10}$$

$$= 1 - e^{-\frac{10}{63}} = \boxed{.1719}$$

Exercise

Determine a such that the Prob. of an earthquake within next a minutes is .99, i.e.

$$P(0 \le X \le a) = .99$$

answer

$$a = 53 \ln(100) = 244.07 \text{ min}$$

$$= 4.068 \text{ hrs}$$

3.2 Cumulative Distribution Functions

Defn

Let X be a r.v. The cumulative distribution function (CDF) of X is

$$F_X(x) = P(X \le x)$$

note

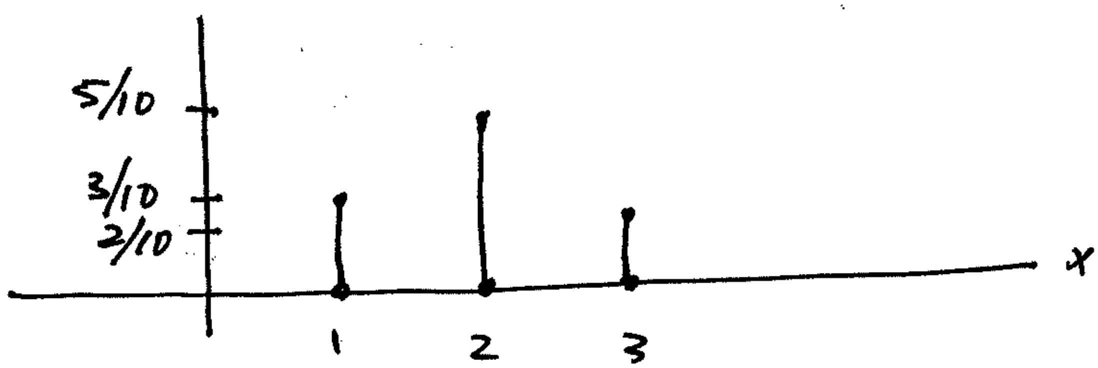
CDF unifies both categories of r.v.s, discrete & continuous.

$$F_X(x) = \begin{cases} \sum_{k \le x} p_X(k) & X \text{ discrete } (\text{range}(X) \subseteq \mathbb{Z}) \\ \int_{-\infty}^{x} f_X(t)\,dt & X \text{ continuous} \end{cases}$$

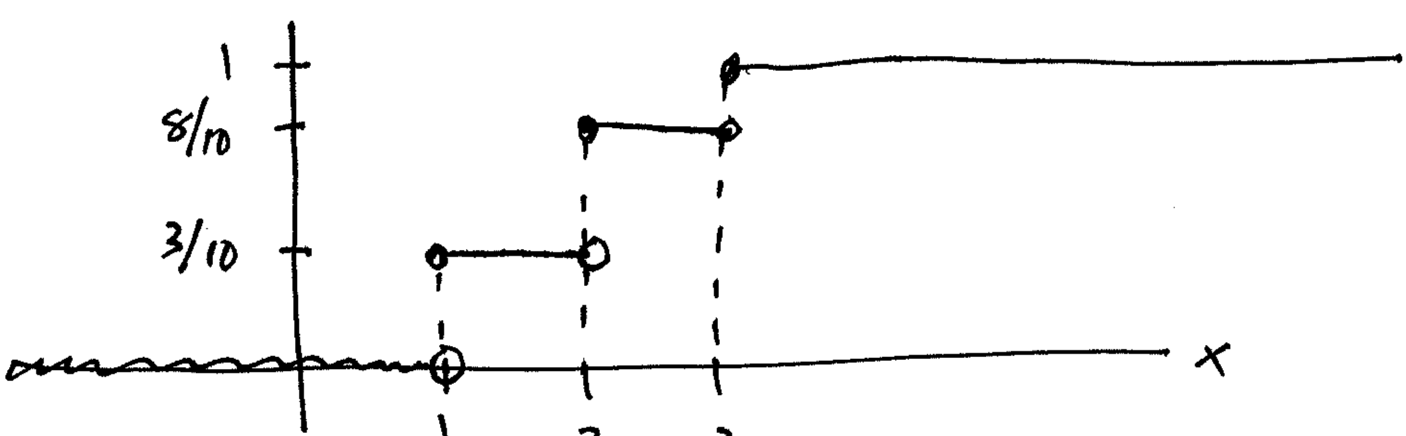
The CDF accumulates probability 'mass' up to x.

Ex (discrete)

PMF: $$p_X(k) = \begin{cases} 3/10 & \text{if } k=1 \\ 5/10 & \text{if } k=2 \\ 2/10 & \text{if } k=3 \\ 0 & \text{otherwise} \end{cases}$$



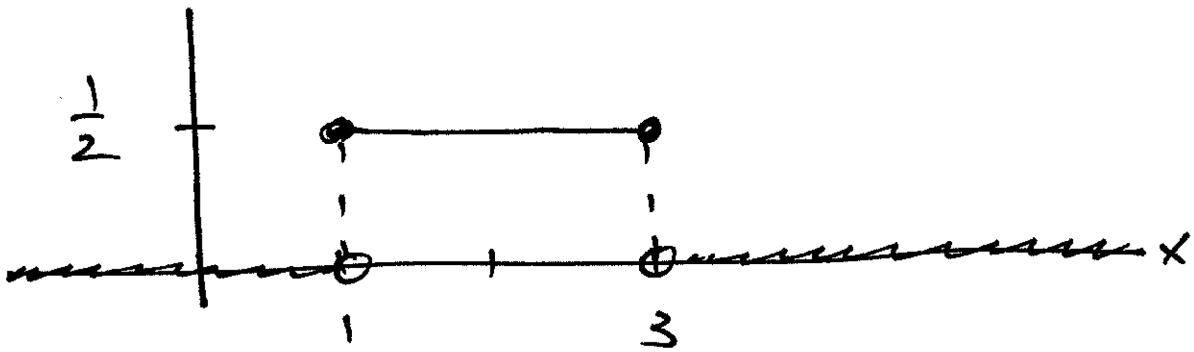
CDF: $$F_X(x) = \begin{cases} 0 & x<1 \\ 3/10 & 1 \le x < 2 \\ 8/10 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$



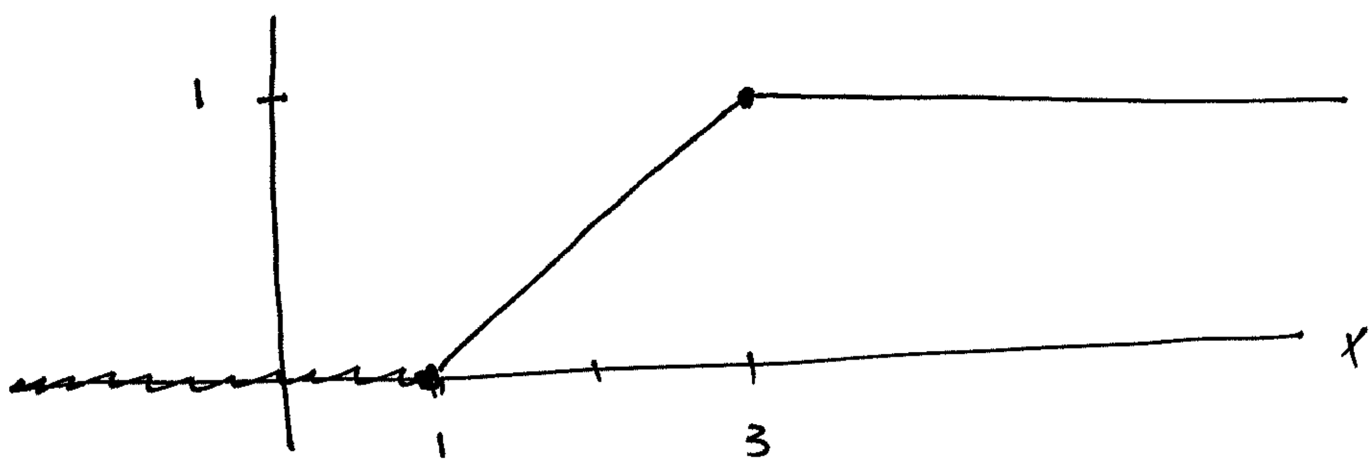
Ex (Continuous)

Uniform on [1, 3]

PDF: $f_X(x) = \begin{cases} 1/2 & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$



CDF: $F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{x-1}{2} & 1 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$



note CDF is increasing

$$F_X(x) = P(X \le x)$$

i.e. $x_1 \le x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$

also if X is discrete, CDF is piecewise constant.

$$F_X(k) = \sum_{i \le k} p_X(i)$$

and

$$\begin{aligned} p_X(k) &= P(X = k) \\ &= P(X \le k) - P(X \le k-1) \\ &= F_X(k) - F_X(k-1) \end{aligned}$$

(assuming X has integer values).

also if X is continuous, CDF is a continuous function

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

Thus

$$f_X(x) = \frac{d}{dx} F_X(x)$$