CSE 107 1-9-26

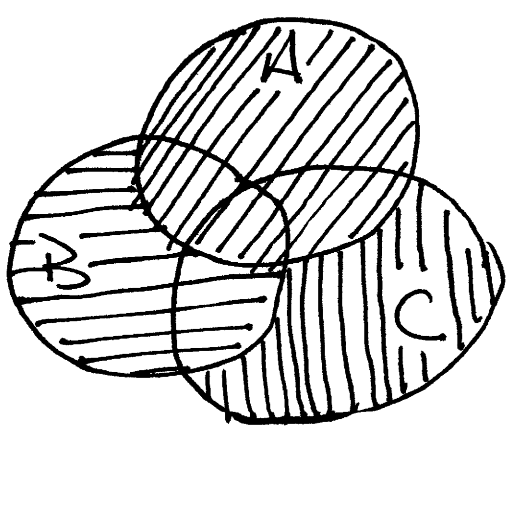
Supplemental Lecture

continuing ... Properties of $P(\cdot)$

(c) follows from (b)

(d) says

$$P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$$



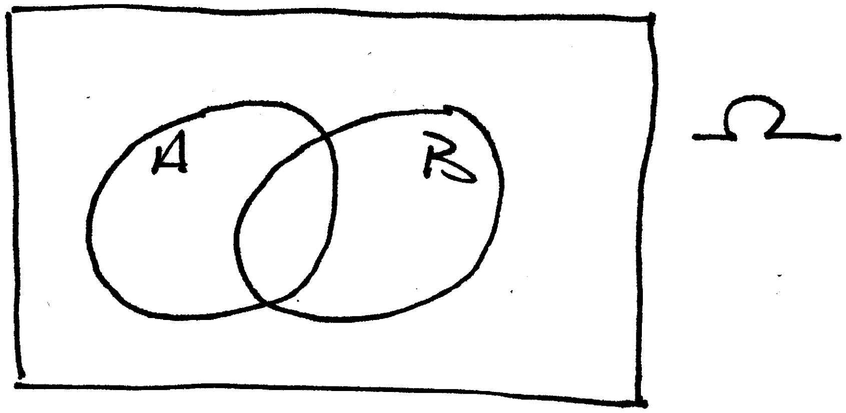
Thus

- A, $A^c \cap B$, $A^c \cap B^c \cap C$ are pairwise disjoint
- $A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C) = A \cup B \cup C$

Result follows from axiom (iii). $\blacksquare$

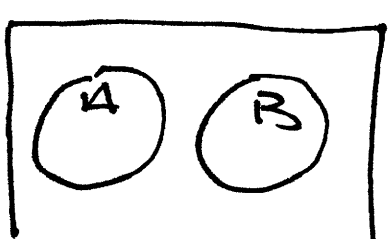
1.3 Conditional Probability

Let $A, B \subseteq \Omega$



How does the information that B occurs affect the probability that A occurs?

Two extreme cases

① 

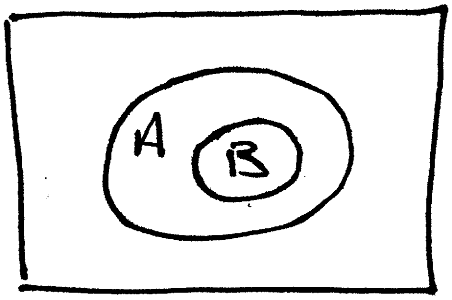
$A \cap B = \phi$

i.e. $A \subseteq B^c$

$B \subseteq A^c$

B occurs $\Rightarrow$ A does not occur

(2)



$B \subseteq A$

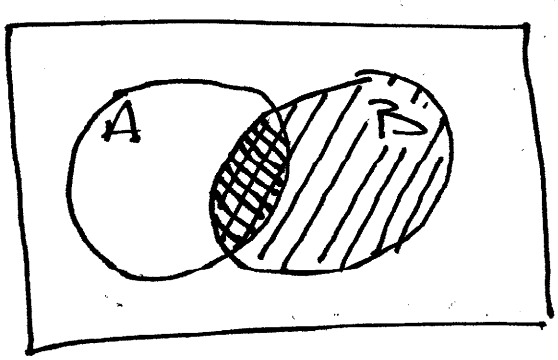
B occurs $\Rightarrow$ A <u>does</u> occur

<u>Defn</u>

Assume $P(B) > 0$. The <u>conditional</u> <u>Probability of A given B</u> is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

<u>Picture</u>



Ω

<u>observe</u>: $P(\cdot\,|B)$ is itself a valid probability law, in that it satisfies the axioms.

<u>Proof</u>

(i) $P(A|B) = \dfrac{P(A\cap B)}{P(B)} \geq 0$ ✓

(ii) $P(\Omega|B) = \dfrac{P(\Omega\cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$ ✓

(iii) If $A_1\cap A_2 = \phi$, then

$$P(A_1\cup A_2|B) = \frac{P((A_1\cup A_2)\cap B)}{P(B)}$$

$$= \frac{P((A_1\cap B)\cup(A_2\cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} \quad \left\{ \begin{array}{l} \text{since} \\ A_1 \cap B \text{ and} \\ A_2 \cap B \text{ are} \\ \text{disjoint} \end{array} \right.$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1|B) + P(A_2|B)$$

$\therefore$ axiom (iii) holds for $P(\cdot|B)$ ✓

$\blacksquare$

Have properties previously proved.

<u>Ex.</u>

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$$

Ex.

let Ω be finite, $P(\cdot)$ discrete uniform

$$P(A) = \frac{|A|}{|\Omega|}$$

Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B| / |\Omega|}{|B| / |\Omega|}$$

$$= \frac{|A \cap B|}{|B|}$$

note! $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ gives

- $P(A \cap B) = P(A|B) \cdot P(B)$

and

- $P(A \cap B) = P(B|A) \cdot P(A)$

Ex.

A test for a certain disease returns a positive result with probability .99, if the subject has the disease; and with probability .10, if the subject does not have the disease. The probability that a random individual has this disease is .05.

- what is the probability of a false positive (positive test and no disease)?

- what is the probability of a false negative (negative test and has disease)?

Let

$T = \{\text{test is positive}\}$

$D = \{\text{subject has disease}\}$

We are given

$P(T|D) = .99 \quad \therefore P(T^c|D) = .01$

$P(T|D^c) = .10 \quad \therefore P(T^c|D^c) = .90$

$P(D) = .05 \quad \therefore P(D^c) = .95$

We must find

$P(T \cap D^c)$ (false positive)

$P(T^c \cap D)$ (false negative)

Thus

$$P(T \cap D^c) = P(T|D^c) \cdot P(D^c)$$

$$= (.10)(.95) = \boxed{.095}$$

and

$$P(T^c \cap D) = P(T^c|D) \cdot P(D)$$

$$= (.01)(.05) = \boxed{.0005}$$

Recall

$$P(A|B)\cdot P(B) = \underbrace{P(A\cap B) = P(A)\cdot P(B|A)}_{\text{Generalize this}}$$

(1) $P(A\cap B) = P(A)\cdot P(B|A)$ ✓

(2) $P(A\cap B\cap C) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)$

(3) $P(A\cap B\cap C\cap D)$

$$= P(A)\cdot P(B|A)\cdot P(C|A\cap B)\cdot P(D|A\cap B\cap C)$$

<u>Proof of (2)</u>

$$RHS = \cancel{P(A)}\cdot \frac{\cancel{P(A\cap B)}}{\cancel{P(A)}}\cdot \frac{P(A\cap B\cap C)}{\cancel{P(A\cap B)}} = LHS.$$

$\blacksquare$

<u>Exercise</u> Prove (3)

Multiplication rule

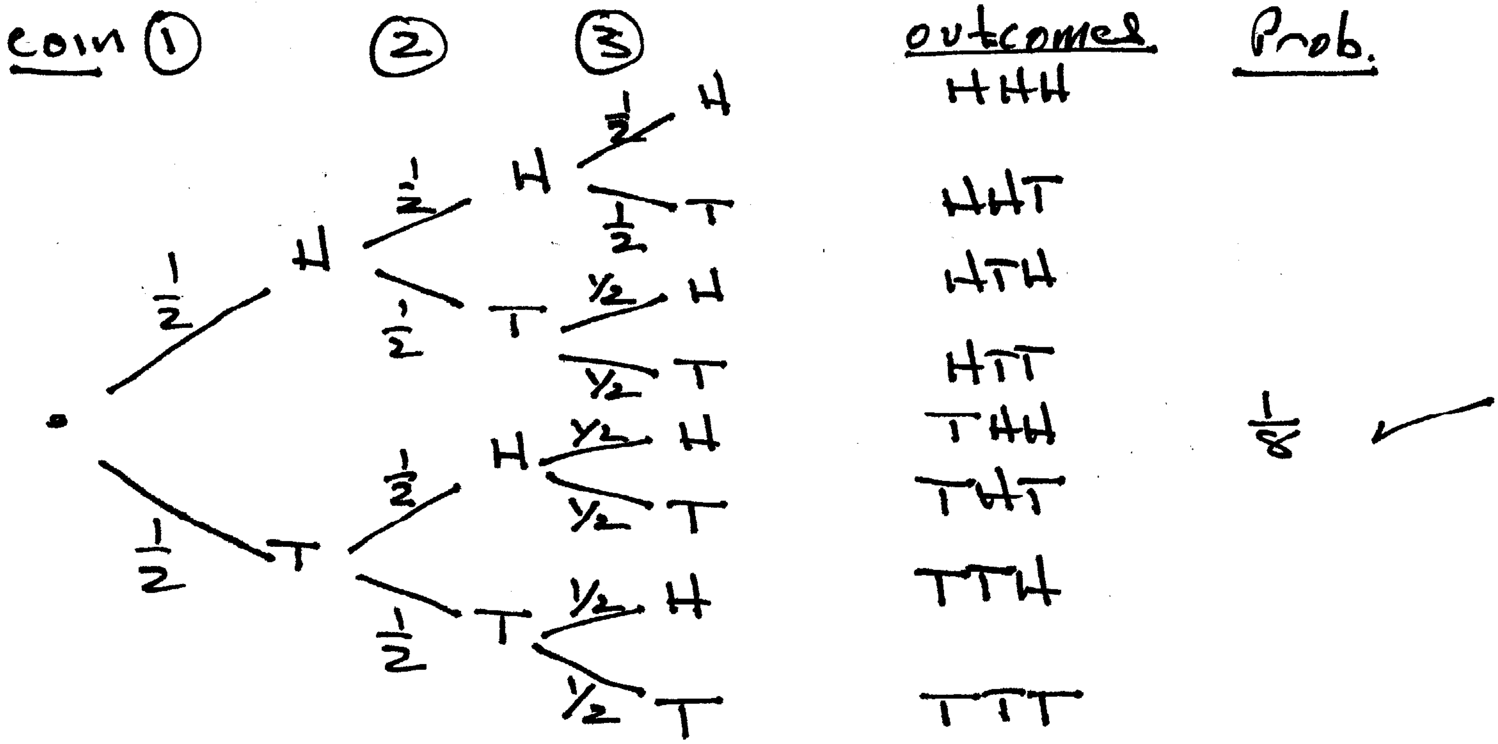
$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2)\cdots$$

$$\cdots\cdots P\left(A_n \,\middle|\, \bigcap_{i=1}^{n-1} A_i\right)$$

i.e.

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1} A_j\right)$$

Ex (again) flip 3 fair coins



$P(T_{①} \cap H_{②} \cap H_{③})$

$= P(T_{①}) \cdot P(H_{②} \mid T_{①}) \cdot P(H_{③} \mid T_{①} \cap H_{②})$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

Defn

A, B are called independent iff

$$P(A|B) = P(A)$$

equivalently

$$P(B|A) = P(B)$$

coins independent means

$$P(H_{2}|T_{1}) = P(H_{2}) = \frac{1}{2}$$

hw1

Prob #2 : Players: $1<2<3$, $\boxed{\text{Play in order } \boxed{1,2,3}}$

$$\left.\begin{array}{lll} & & \text{Prob} \\ A = \{\text{win against } 1\} & & P_1 \\ B = \{\text{win against } 2\} & & P_2 \\ C = \{\text{win against } 3\} & & P_3 \end{array}\right\} \quad P_1 > P_2 > P_3$$

$$E = \{\text{Win tournament}\}$$
$$= (A \cap B) \cup (B \cap C)$$

$$P(A \cap B) = P(A) \cdot \underbrace{P(B|A)}_{\substack{\text{assume} \\ = P(B)}} = P(A) \cdot P(B) = P_1 P_2$$

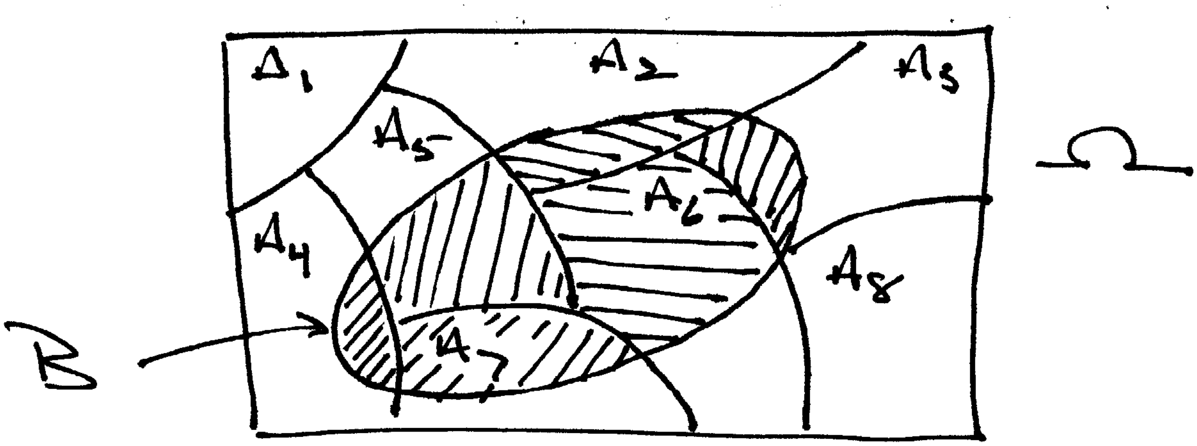
$$P(B \cap C) = P(B) \cdot \underline{P(C|B)} = P(B) \cdot P(C) = P_2 P_3$$

$$P(A \cap B \cap C) = P(A) \cdot \underbrace{P(B|A)}_{P(B)} \cdot \underbrace{P(C|B \cap A)}_{P(C)} = P_1 P_2 P_3$$

1.4 Total Probability Theorem and Baye's Rule

Let $A_1, A_2, \ldots, A_n$ be a Partition of Ω, i.e.

- Pairwise disjoint: $A_i \cap A_j = \phi$ if $i \neq j$
- $\Omega = A_1 \cup A_2 \cup \cdots \cup A_n$
- $P(A_i) > 0$ for $i = 1, 2, \ldots, n$



Let $B \subseteq \Omega$.

Total Probability Theorem

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \cdots$$

$$\cdots + P(A_n) \cdot P(B|A_n)$$

Proof

1^{st} eqn. follows from axiom (iii).

2^{nd} eqn. " " definition of conditional Probability.

∎

Ex. $\Omega = \{1,2,3,4,5,6\}$

Roll a 6-sided die. If result is in $\{1,2,3\}$, roll again, otherwise stop. What is the probability that sum of all rolls is ≥ 5?

Solution

Let $A_i = \{1^{st}$ roll is $i\}$ $(i = 1, 2, \ldots, 6)$, so $P(A_i) = \frac{1}{6}$. Let $B = \{\text{sum} \geq 5\}$

note

If A_1 occurs, then B occurs iff 2^{nd} roll $\in \{4,5,6\}$

" A_2 " " " " " " " $\in \{3,4,5,6\}$

" A_3 " " " " " " " $\in \{2,3,4,5,6\}$

Thus

$$P(B|A_1) = \frac{3}{6} = \frac{1}{2}$$

$$P(B|A_2) = \frac{4}{6} = \frac{2}{3}$$

$$P(B|A_3) = \frac{5}{6}$$

$$P(B|A_4) = 0$$

$$P(B|A_5) = 1$$

$$P(B|A_6) = 1$$

Hence

$$P(B) = P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + \dots$$

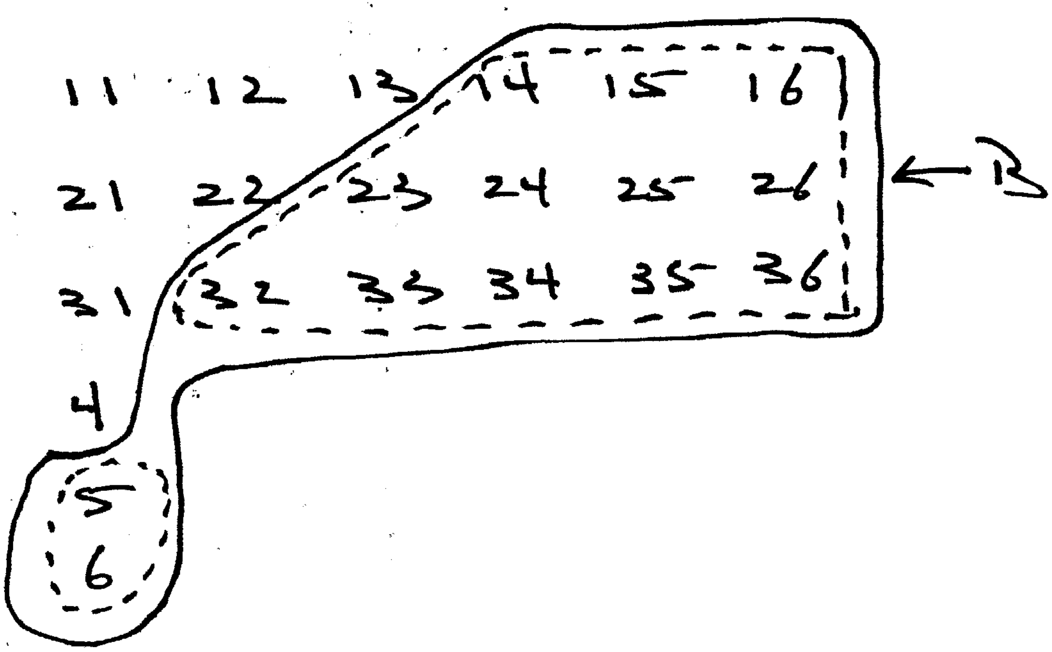
$$\dots + P(A_6)\cdot P(B|A_6)$$

$$= \frac{1}{6}\cdot\frac{3}{6} + \frac{1}{6}\cdot\frac{4}{6} + \frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot 0 + \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 1$$

$$= \frac{1}{6}\left(\frac{3+4+5+0+6+6}{6}\right) = \frac{1}{6}\cdot\frac{24}{6} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

Alternate solution

Ω can be pictured as



$$P(B) = \frac{12}{36} + \frac{2}{6} = \cdots = \boxed{\frac{2}{3}}$$

Baye's Rule

Again let $A_1, A_2, \ldots, A_n$ be a Partition of Ω, and $P(A_i) > 0$ for $i = 1, 2, \ldots, n$. Assume $B \subseteq \Omega$ with $P(B) > 0$. Then for each i, we have

$$P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(B)}$$

$$= \frac{P(A_i) \cdot P(B \mid A_i)}{P(A_1) \cdot P(B \mid A_1) + \cdots\cdots + P(A_n) \cdot P(B \mid A_n)}$$

Proof

Recall $P(B)\cdot P(A|B) = P(A\cap B) = P(A)\cdot P(B|A)$

Thus $$P(A|B) = \frac{P(A)\cdot P(B|A)}{P(B)},$$

so 1^{st} eqn. holds. 2^{nd} eqn. follows from total Probability ∎

note: Bayes is used for 'inference' Problems. say have causes

$$A_1, A_2, \dots A_n$$

that may cause the effect B.

$$\begin{array}{ccc} A_1 & \searrow & \\ A_2 & \longrightarrow & B \\ \vdots & & \\ A_n & \nearrow & \\ \text{causes} & & \text{effect} \end{array}$$

The cond. Prob. $P(B|A_i)$ is a model of cause-effect strength.

$$A_i \xrightarrow{\hspace{3em}} B$$

$P(B|A_i)$
strength
of causation

what is $P(A_i|B)$?

$$A_i \xleftarrow{\hspace{3em}} B$$

$P(A_i|B)$ is strength of inference A_i given B.

Prior, likelihood

$$P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(B)}$$

Posterior, marginal

Ex. Disease example again.

$T = \{\text{Test Positive}\}$

$D = \{\text{has disease}\}$

$P(T \mid D) = .99$

$P(T \mid D^c) = .10$

$P(D) = .05$, $P(D^c) = .95$

Find $P(D|T)$

$$P(D|T) = \frac{P(D)\cdot P(T|D)}{P(T)}$$

Partition: $T = (T \cap \underset{\substack{\uparrow \\ D}}{D}) \cup (T \cap \underset{\substack{\uparrow \\ D^c}}{D^c})$

Thus

$$P(D|T) = \frac{P(D)\cdot P(T|D)}{P(D)\cdot P(T|D) + P(D^c)\cdot P(T|D^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99) + (.95)(.10)}$$

$$= \cdots = .34256\ldots \approx \boxed{.3426}$$