CSE 107 4-28-26

<u>Defn</u>

A random variable $X : \Omega \to \mathbb{R}$ is called continuous iff there exists a function

$$f_X : \mathbb{R} \to \mathbb{R}$$

called the <u>Probability Density Function</u> (PDF) such that

$$P(X \in I) = \int_I f_X(x)\,dx$$

for 'any' $I \subseteq \mathbb{R}$. In particular if $I = [a,b]$, then

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

<u>Note</u>: integrating over $[a,b]$ $(a,b]$, $[a,b)$ or (a,b) are the same:

$$P(X=a) = \int_a^a f_X(x)\,dx = 0$$

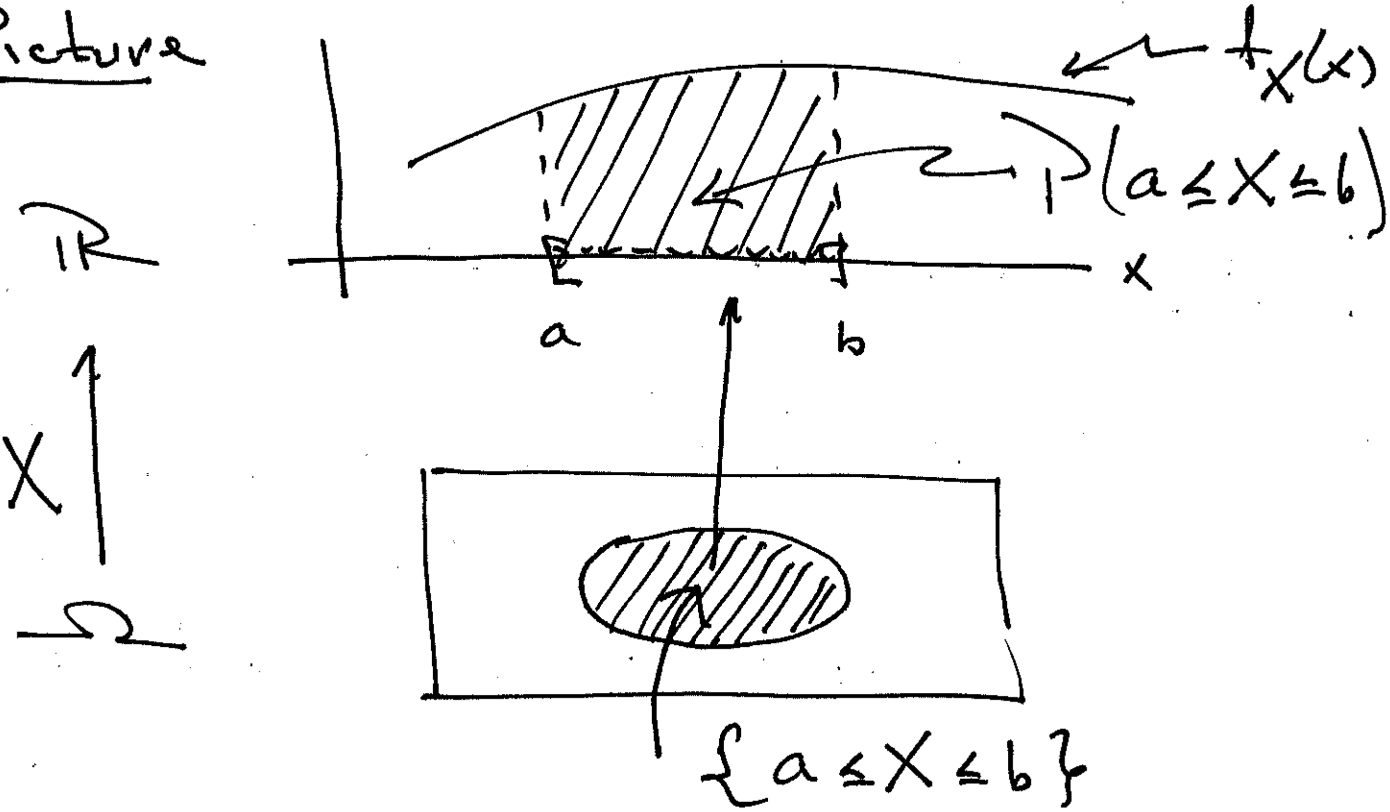
Note

To be a valid PDF we must have

- $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
- $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = P(X \in \mathbb{R})$

$$= P(\Omega) = 1$$

Picture



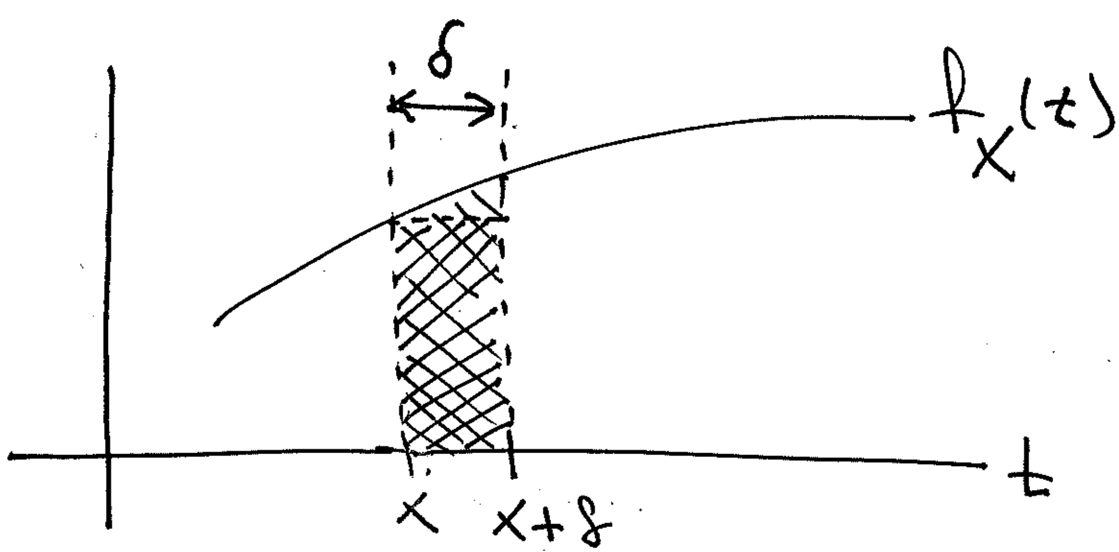
Note

Unlike the PMF, the PDF does not give probability (or 'mass') at a point.

Instead, $f_X(x)$ gives the 'mass density' at x, i.e. mass per unit length.

Let $\delta > 0$ be a small real number.

Then $f_X(t)$ is approx. constant on the interval $[x, x+\delta]$ (assuming f_X is continuous).



$$\underbrace{P(x \le X \le x+\delta)}_{\text{Probability i.e. 'mass'}} = \int_x^{x+\delta} f_X(t)\,dt \approx \underbrace{f_X(x)}\cdot\underbrace{\delta}_{\text{length}}$$

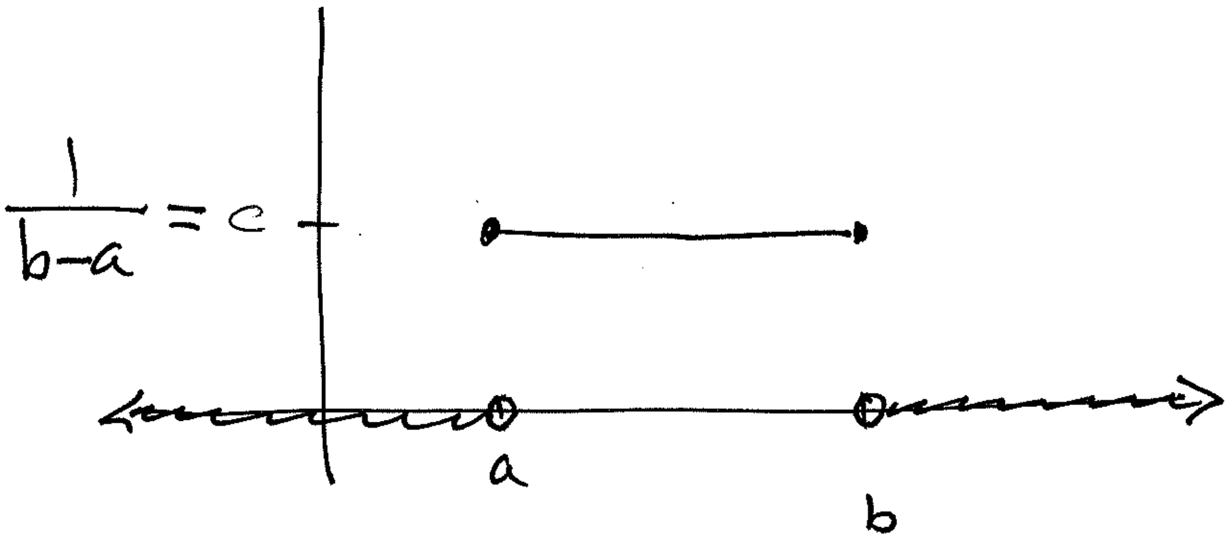
Thus $f_X(x)$ has units

$$\frac{\text{'mass'}}{\text{length}} = \text{density}.$$

<u>note</u>: f_X may have some jump discontinuities.

<u>Ex.</u> Continuous uniform r.v. on $[a, b]$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



The const c is determined by

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

so

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b c\,dx = cx\Big|_a^b = c(b-a)$$

$$\therefore\; c = \frac{1}{b-a}$$

Ex Piecewise Constant PDF.

Alice's driving time to work is 40–44 min. in good weather and 44–50 min in bad, uniformly distributed in each case. Suppose P(good weather) = $\frac{2}{3}$. Determine the PDF of X, Alice's driving time.

<u>Solution</u> We are given

$$f_X(x) = \begin{cases} c_1 & \text{if } 40 \le x \le 44 \text{ (good)} \\ c_2 & \text{if } 44 < x \le 50 \text{ (bad)} \\ 0 & \text{otherwise} \end{cases}$$

note

$$\frac{2}{3} = P(\text{good}) = \int_{40}^{44} c_1\, dx = c_1 \cdot x \Big|_{40}^{44} = 4c_1$$

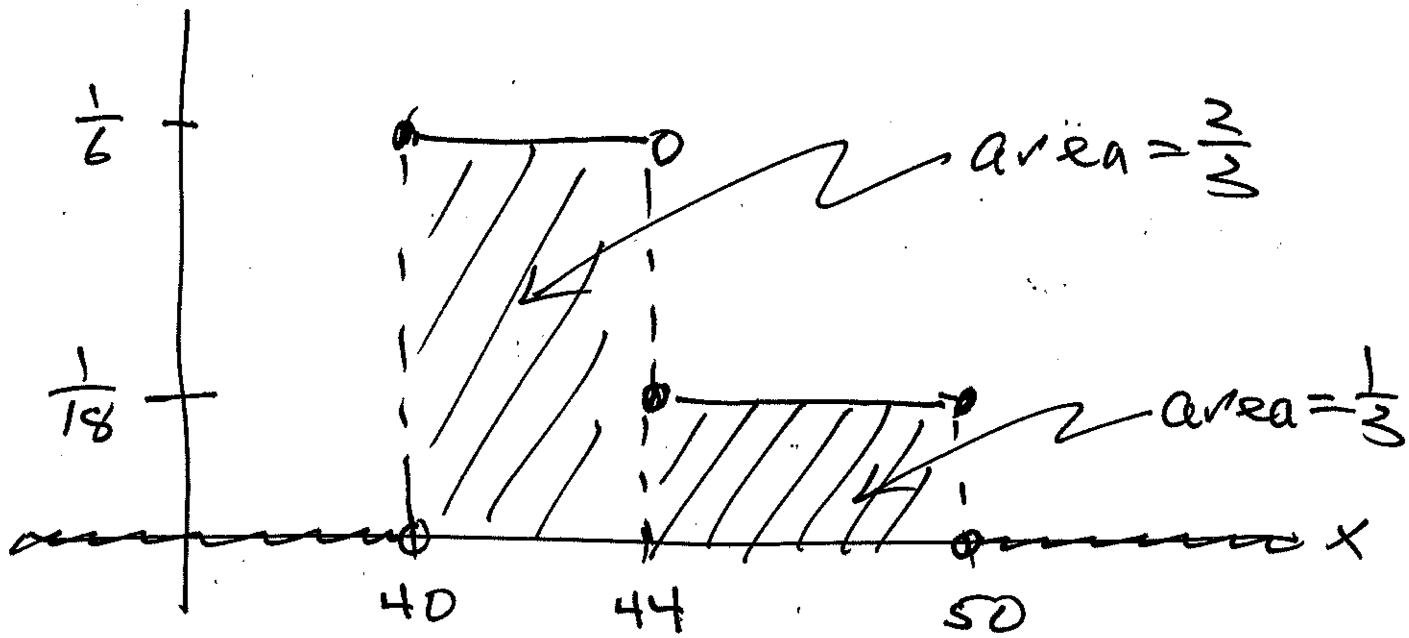
$$\therefore \quad c_1 = \frac{1}{4} \cdot \frac{2}{3} = \boxed{\frac{1}{6}}$$

also

$$\frac{1}{3} = P(\text{bad}) = \int_{44}^{50} c_2\, dx = 6 \cdot c_2$$

$$\therefore \quad c_2 = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18}.$$

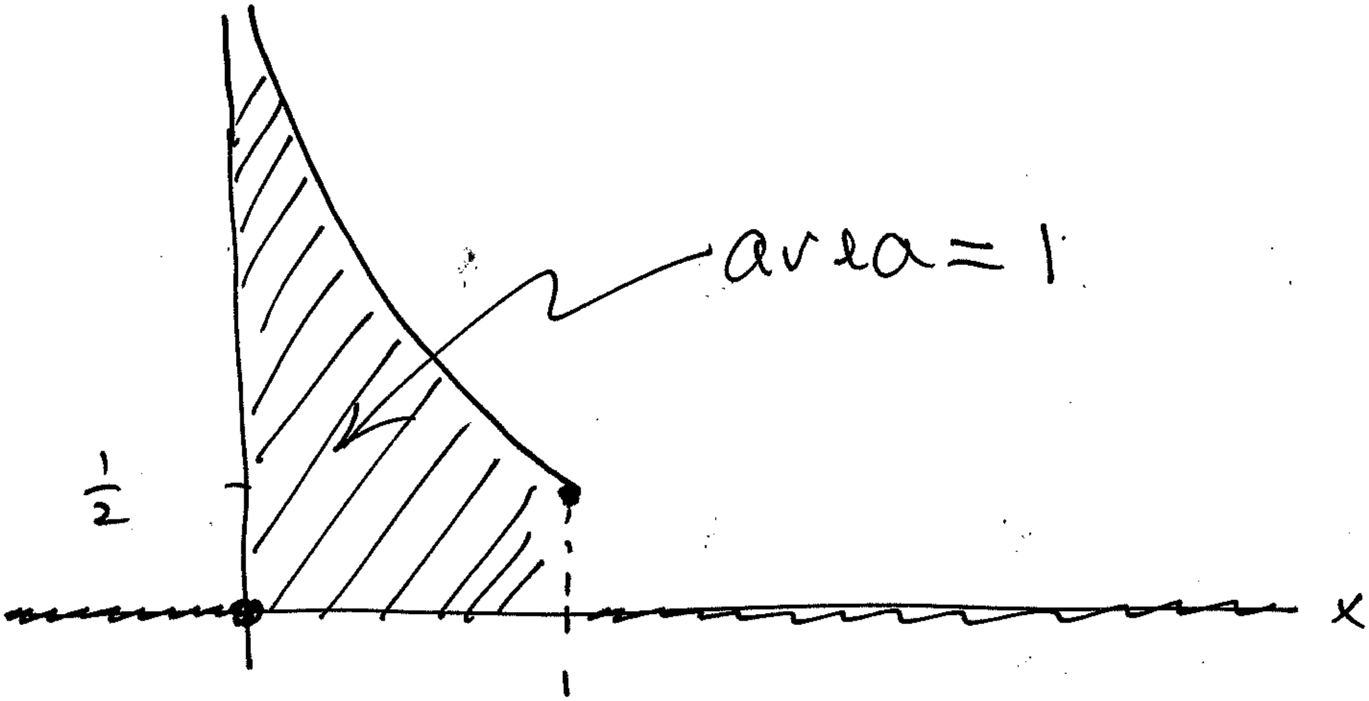
$$\therefore \quad f_X(x) = \begin{cases} 1/6 & \text{if } 40 \le x \le 44 \\ 1/18 & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$



A PDF need not be bounded above.

<u>Ex.</u>

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \tfrac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$= \cancel{\tfrac{1}{2}}\,\frac{x^{\frac{1}{2}}}{\cancel{\tfrac{1}{2}}}\Bigg|_0^1 = \sqrt{1} - \sqrt{0} = \boxed{1}$$

actually

$$\int_0^1 \tfrac{1}{2}x^{-\frac{1}{2}}\,dx = \lim_{a\to 0^+} \int_a^1 \tfrac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$= \lim_{a\to 0^+} \left(x^{\frac{1}{2}}\right)\Big|_a^1 = \lim_{a\to 0^+}\left(\sqrt{1} - \sqrt{a}\right)$$

$$= \boxed{1}$$

<u>Defn</u>

The <u>expected value</u> of a continuous r.v. X is given by

$$\boxed{E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx}$$

<u>Expected value rule</u>

let X be a cont. r.v., $g: \mathbb{R} \to \mathbb{R}$, then

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx}$$

The n^{th} moment of X is

$$E[X^n]$$

Defn

The variance of X is

$$\boxed{Var(X) = E[(X-E[X])^2]}$$

Theorem (Var in terms of moments)

$$\boxed{Var(X) = E[X^2] - E[X]^2}$$

<u>Proof</u>. Let $\mu = E[X]$.

$$Var(X) = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f_X(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{\infty} x^2 f_X(x)\,dx}_{E[X^2]} - 2\mu \underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx}_{\substack{E[X] \\ \mu}} + \mu^2 \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{1}$$

$$= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2$$

$$= E[X^2] - E[X]^2$$

∎

Let $Y = aX + b$ where $a, b \in \mathbb{R}$ and X is a cont. r.v.

Then

- $E[Y] = aE[X] + b$
- $Var(Y) = a^2 Var(X)$

<u>Exercise</u>

Prove these.

Ex. Cont. unif. on $[a,b]$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_a^b x \cdot \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \cdot \frac{1}{2} x^2 \Big|_a^b = \frac{1}{b-a} \cdot \frac{1}{2}(b^2 - a^2)$$

$$= \frac{1}{2} \cdot \frac{\cancel{(b-a)}(b+a)}{\cancel{b-a}} = \boxed{\frac{a+b}{2}}$$

$$E[X^2] = \int_a^b x^2 \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \cdot \frac{1}{3} x^3 = \frac{1}{b-a} \cdot \frac{1}{3} (b^3 - a^3)$$

$$= \frac{1}{3} \frac{\cancel{(b-a)}(b^2+ab+a^2)}{\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}$$

Thus

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \cdots = \boxed{\frac{(b-a)^2}{12}}$$