CSE 107 4-21-26

- mid 1 : Th 4/23

Mean and Variance of Geometric(p)

Recall Geometric(p) has PMF

$$p_X(k) = \begin{cases} (1-p)^{k-1} p & k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p$$

$$E[X^2] = \sum_{k=1}^{\infty} k^2(1-p)^{k-1} \cdot p$$

and

$$Var(X) = E[X^2] - E[X]^2$$

Another way: Define

$$A_1 = \{X=1\} = \{1^{st} \text{ flip head}\}$$

and

$$A_2 = \{X>1\} = \{1^{st} \text{ flip tails}\}$$

Then $P(A_1) = p$, $P(A_2) = 1-p$.

Also note

$$E[X \mid A_1] = 1$$

and

$$E[X \mid A_2] = 1 + E[X]$$

Thus, by total expectation thm

$$E[X] = E[X \mid A_1] \cdot P(A_1) + E[X \mid A_2] \cdot P(A_2)$$

$$E[X] = 1 \cdot p + (1 + E[X]) \cdot (1-p)$$

$\vdots$ solve for $E[X]$

$$E[X] = \boxed{\frac{1}{p}}$$

Note that

$$E[X^2 \mid A_1] = 1$$

and

$$E[X^2 \mid A_2] = E[(1+X)^2]$$

$$= E[1 + 2X + X^2]$$

$$= 1 + 2E[X] + E[X^2]$$

$$= 1 + \frac{2}{p} + E[X^2]$$

So

$$E[X^2] = p + \left(1 + \frac{2}{p} + E[X^2]\right)\cdot(1-p)$$

solve for $E[X^2]$

$$E[X^2] = \frac{2-p}{p^2}$$

Thus

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$= \boxed{\frac{1-p}{p^2}}$$

2.7 Independence

Defn

We say r.v.s X, Y are **independent** iff for all $x \in \text{range}(X)$ and $y \in \text{range}(Y)$, the events

$$\{X = x\} \text{ and } \{Y = y\}$$

are independent, i.e.

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

i.e.

$$\boxed{p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)}$$

note if $p_Y(y) > 0$ for $y \in \text{range}(Y)$,

this is equiv. to

$$\frac{p_{X,Y}(x,y)}{p_Y(y)} = p_X(x)$$

i.e.

$$\boxed{p_{X|Y}(x|y) = p_X(x)}$$

As usual, can extend to a conditional prob. law

$$P(\cdot\,|A) \text{ provided } P(A) > 0$$

Defn

We say r.v.s X, Y are conditionally indep. given A iff

$$P(X=x, Y=y \mid A) = P(X=x \mid A) \cdot P(Y=y \mid A)$$

for all $(x,y) \in \text{range}(X) \times \text{range}(Y)$

i.e.

$$\boxed{P_{X,Y|A}(x,y) = P_{X|A}(x) \cdot P_{Y|A}(y)}$$

Warning

$$\boxed{\text{cond. indep. of } X, Y} \;\not\Rightarrow\; \boxed{\text{indep. of } X, Y}$$

$$\boxed{\text{cond. indep. of } X, Y} \;\not\Leftarrow\; \boxed{\text{indep. of } X, Y}$$

see fig. 2.15 on p. 111.

Consequence of indep. of X, Y

Theorem

if X, Y are indep., then

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

Proof

$$E[XY] = \sum_x \sum_y xy\, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y xy \cdot p_X(x) \cdot p_Y(y)$$

$$= \sum_x \sum_y (x p_X(x)) \cdot (y p_Y(y))$$

$$= \sum_x \left((x p_X(x)) \cdot \sum_y (y p_Y(y)) \right)$$

$$= \left(\sum_x x p_X(x) \right) \cdot \left(\sum_y y p_Y(y) \right)$$

$$= E[X] \cdot E[Y]$$

∎

Theorem

If X, Y are indep., then so are $g(X)$ and $h(Y)$, for any fcns. g and h.

(see Prob. #44 on p. 134)

Thus if X, Y are indep.

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)].$$

Another consequence!

Theorem

If X, Y are indep., then

$$\boxed{Var(X+Y) = Var(X) + Var(Y)}$$

Proof

Let X, Y be indep. r.v.s. Set

$$U = X - E[X]$$

and

$$V = Y - E[Y]$$

Then

$$E[U] = E[X] - E[X] = 0$$

$$E[V] = \quad .. \qquad .. \quad = 0$$

also

$$U + V = (X - E[X]) + (Y - E[Y])$$

$$= (X+Y) - E[X+Y]$$

Thus

$$Var(X+Y) = E[((X+Y) - E[X+Y])^2]$$

$$= E[(U+V)^2]$$

$$= E[U^2 + 2UV + V^2]$$

$$= E[U^2] + 2E[UV] + E[V^2]$$

$$= E[U^2] + 2\cancel{E[U]}^{0}\cdot\cancel{E[V]}^{0} + E[V^2]$$

$$= E[(X - E[X])^2] + E[(Y - E[Y])^2]$$

$$= Var(X) + Var(Y).$$

Variance of Binomial

Recall if X_i is Bernoulli(p), then

$$X = X_1 + X_2 + \cdots + X_n$$

is Binomial(n,p) iff the X_i are indep. Thus

$$\begin{aligned} Var(X) &= Var(X_1) + \cdots + Var(X_n) \\ &= p(1-p) + \cdots + p(1-p) \\ &= \boxed{np(1-p)} \end{aligned}$$