cse 107 4-2-26

Ex. roll 2 $\underbrace{\text{fair}}$ dice

eq. likely outcomes

$\{max=4\}$

$\{min=4\}$

$\{1^{st}>2^{nd}\}$

$\Omega$

| | | | | | |
|---|---|---|---|---|---|
| 11 | 12 | 13 | 14 | 15 | 16 |
| 21 | 22 | 23 | 24 | 25 | 26 |
| 31 | 32 | 33 | 34 | 35 | 36 |
| 41 | 42 | 43 | 44 | 45 | 46 |
| 51 | 52 | 53 | 54 | 55 | 56 |
| 61 | 62 | 63 | 64 | 65 | 66 |

$$P(\text{sum is even}) = \frac{18}{36} = \frac{1}{2}$$

$$P(\text{sum is odd}) = 1 - \frac{1}{2} = \frac{1}{2}$$

$P(\max = 4) = \frac{7}{36}$

$P(\min = 4) = \frac{5}{36}$

$P(1^{st} > 2^{nd}) = \frac{15}{36}$

<u>Exercise</u>: find

$P(1^{st} = 2^{nd})$

$P(\text{at least one die} = 3)$

$P(\max \geq 4)$

$P(\max \leq 4)$

$P(\min \geq 4)$

$P(\min \leq 4$

## continuous models

Here $\Omega$ is a 'continuum',
like an interval $\subseteq \mathbb{R}$,
region $\subseteq \mathbb{R}^2$, volume $\subseteq \mathbb{R}^3$.

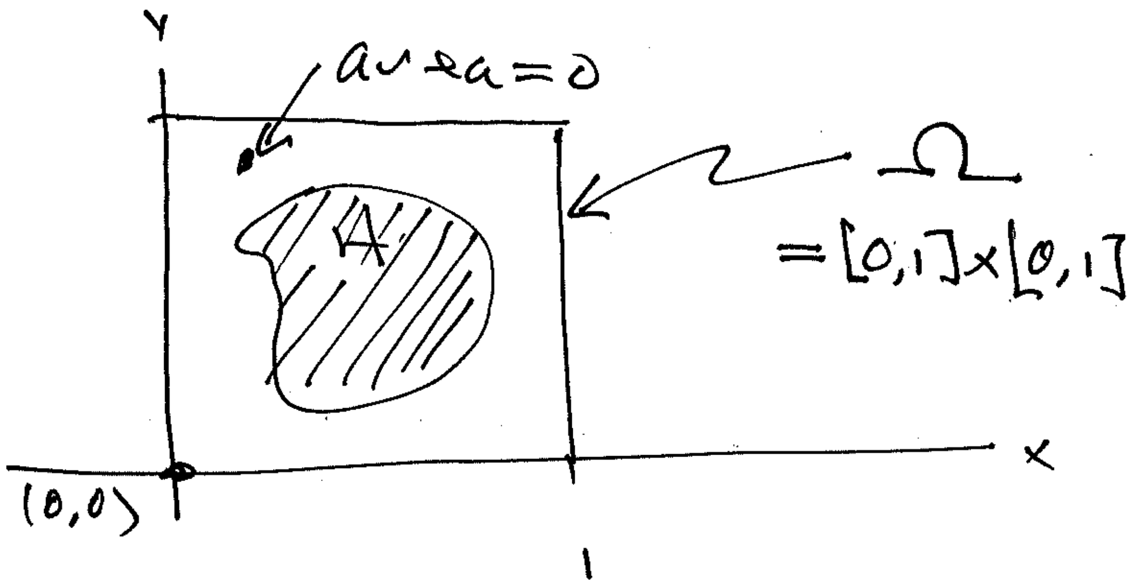
Ex.
R. and J. have a date.
Each will be delayed by a
random time in range 0 to 1
hour. The 1st to arrive
will wait 15 min., then leave.

What is the Prob. that they meet?

Let $x$ be R. delay time, and $y$ be J. " " .

Thus $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Random means all Pairs $(x,y)$ are 'equally likely'.



'equally likely' means $P(A)$ is proportional to $\text{area}(A)$

so

$$P(A) = c \cdot \text{area}(A)$$

note

$$1 = P(\Omega) = c \cdot \text{area}(\Omega)$$

$$\therefore \quad \boxed{c = \frac{1}{\text{area}(\Omega)}}$$

note $P((x,y)) = 0$

In this example $c = 1$, since $\text{area}(\Omega) = 1$.

We seek $P(M)$ where

$$M = \{(x,y) \mid |x-y| \le \tfrac{1}{4},\ (x,y) \in \Omega\}$$

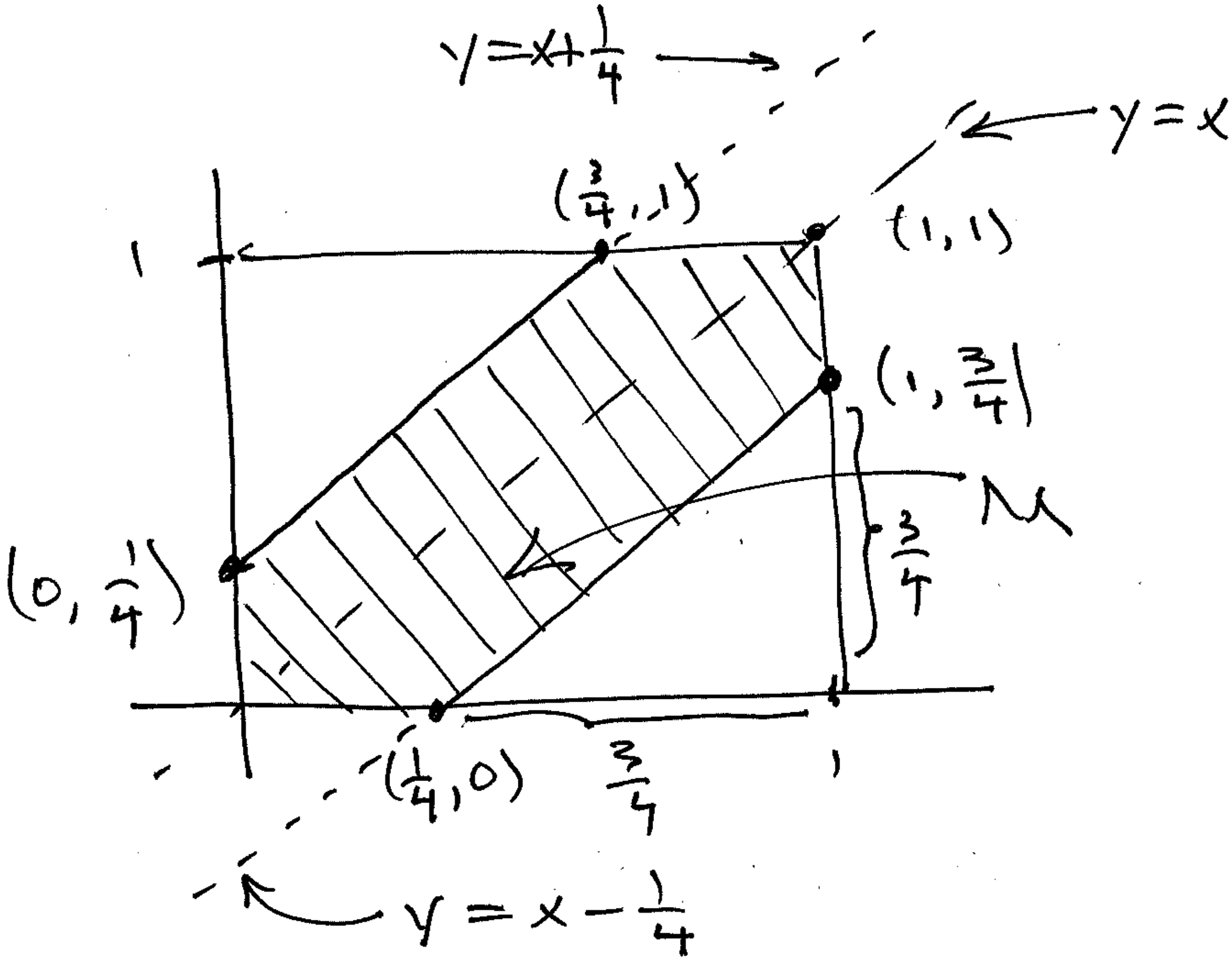
observe $(x,y) \in M$ iff

$$|x-y| \le \tfrac{1}{4}$$

$$\Leftrightarrow \quad -\tfrac{1}{4} \le x-y \le \tfrac{1}{4}$$

$$\Leftrightarrow \quad y \le x+\tfrac{1}{4} \text{ and } y \ge x-\tfrac{1}{4}$$

$\square$




$$P(M) = \text{area}(M)$$

$$= 1 - \text{area}(\nabla) - \text{area}(\Delta)$$

$$= 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16}$$

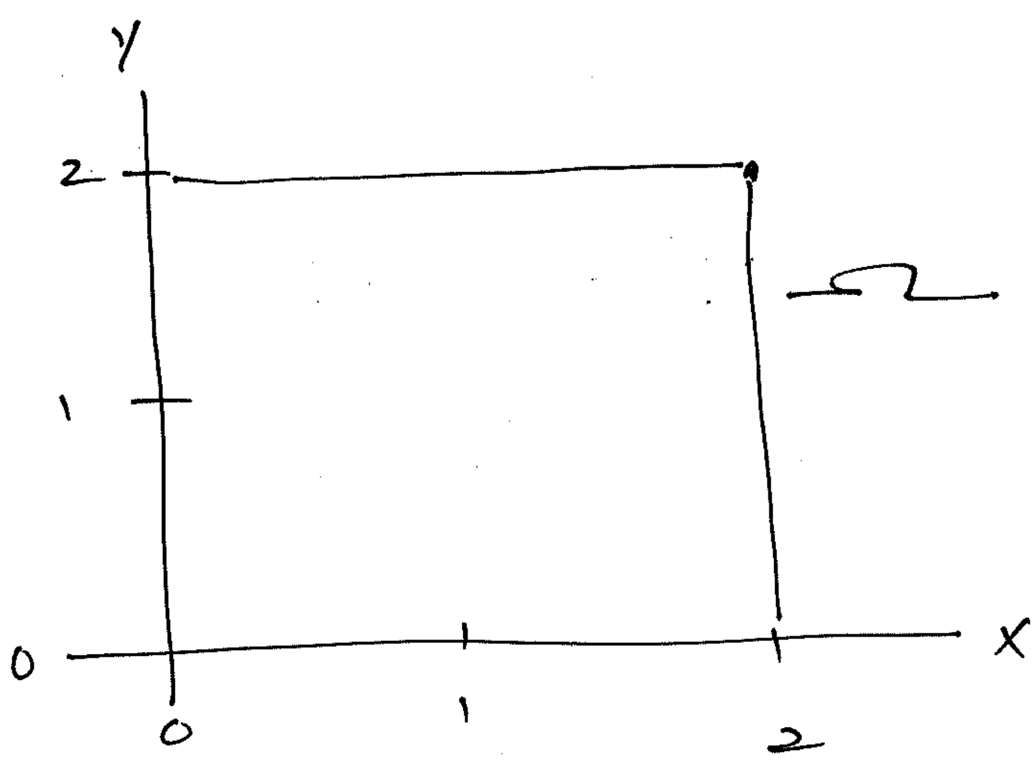
$$= \boxed{\frac{7}{16}}$$

## Exercise

Alice & Bob each pick a random number in $[0,2]$

let Alice's # be $x$, Bob's # $y$.

What is the probability that both $2x > y^2$ and $2y > x^2$?



# Properties of $P(\cdot)$

One can prove:

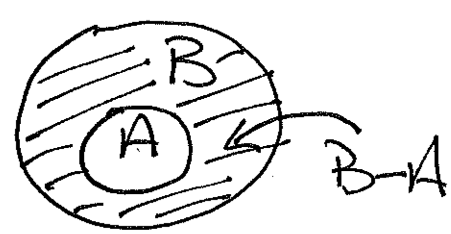
(a) If $A \subseteq B$, then $P(A) \le P(B)$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(c) $P(A \cup B) \le P(A) + P(B)$

(d) $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

## Proof of (a)




$$A \subseteq B \Rightarrow \begin{cases} B = A \cup (B - A) \\ A \cap (B - A) = \phi \end{cases}$$

axiom (iii)

$$\Rightarrow P(B) = P(A) + P(B-A) \geq P(A)$$

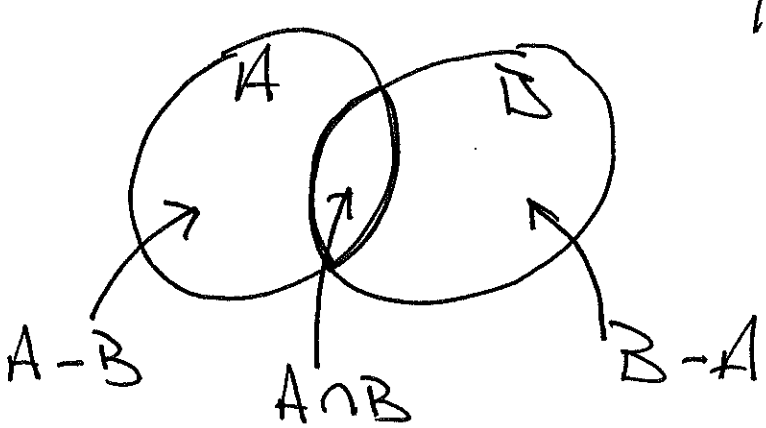
axiom (i)

$$\Rightarrow P(A) \leq P(B) \qquad \blacksquare$$

## Proof of (b)

$$\boxed{P(A\cup B) = P(A-B) + P(A\cap B) + P(B-A)}$$

by axiom (iii)



also by axiom (iii)

$$P(B) = P(B-A) + P(A \cap B)$$

$$\therefore \boxed{P(B-A) = P(B) - P(A \cap B)}$$

also

$$\boxed{P(A-B) = P(A) - P(A \cap B)}$$

so

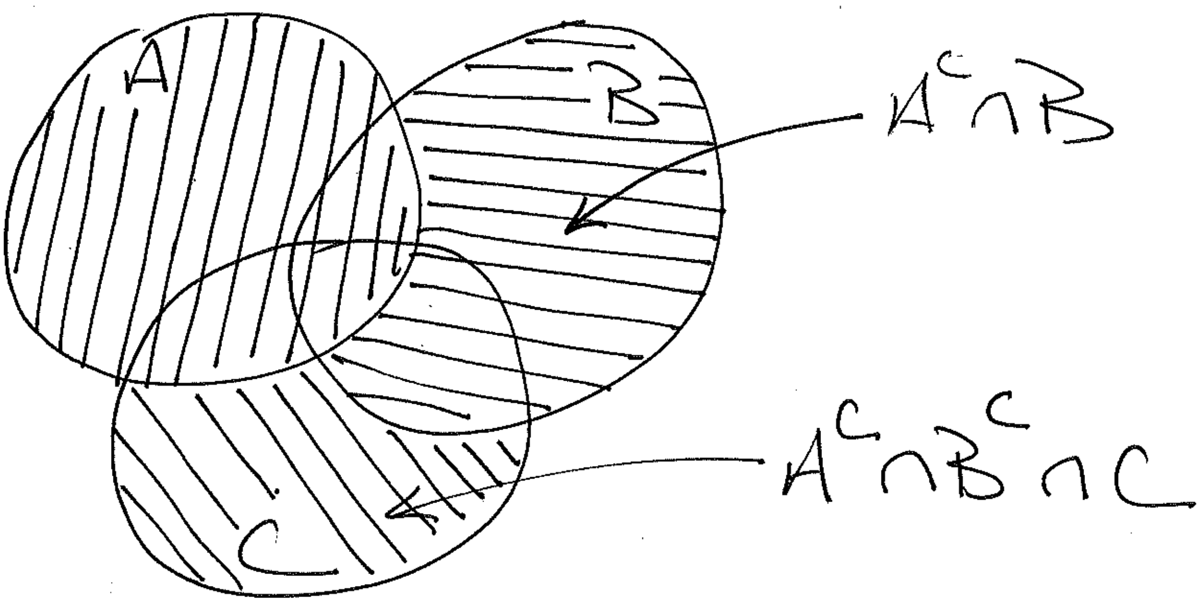
$$P(A \cup B) = P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)}$$
$$+ P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

Proof of (c)

exercise. follows from (b)

Proof of (d)




Observe

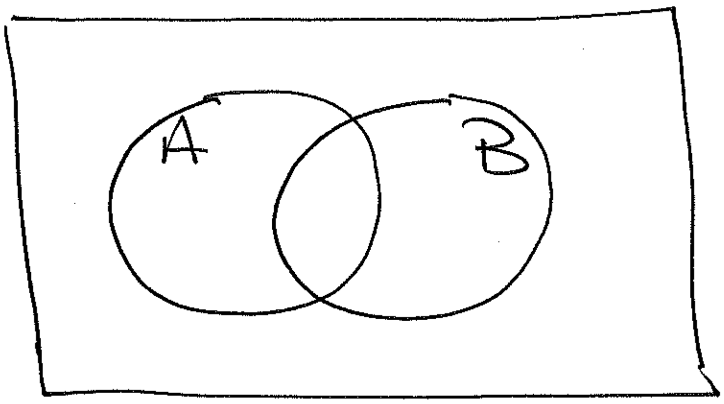
- $A$, $A^c \cap B$, $A^c \cap B^c \cap C$ are pairwise disjoint.
- $A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C) = A \cup B \cup C$

By axiom (iii)

$$P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$$

## 1.3 Conditional Probability

Suppose $A, B \subseteq \Omega$




$\Omega$

Given information that $B$ occurs, in general this affects our knowledge/belief in Prob. that $A$ occured.