CSE 107 4-17-26

Supplemental Lecture

Proof of (2)

$E[X^2] = \lambda^2 + \lambda$ where the r.v. X is Poisson(λ).

$$E[X^2] = \sum_{k=\cancel{0}1}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} (k+1) \cdot \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \left(k \frac{\lambda^k}{k!} + \frac{\lambda^k}{k!} \right)$$

$$= \lambda\left(\underbrace{e^{-\lambda} \sum_{k=\cancel{0}1}^{\infty} k \frac{\lambda^k}{k!}}_{} + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= \lambda\left(\lambda + \underbrace{e^{-\lambda} \cdot e^{\lambda}}_{1} \right)$$ by (*) from Part (1)

$$= \lambda(\lambda + 1) = \lambda^2 + \lambda$$

Thus

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= (\cancel{\lambda^2} + \lambda) - \cancel{\lambda^2}$$

$$= \lambda$$

Thus:
- $E[X] = \lambda$
- $\mathrm{Var}(X) = \lambda$

Ex. Quiz Problem

A game consists of 2 questions

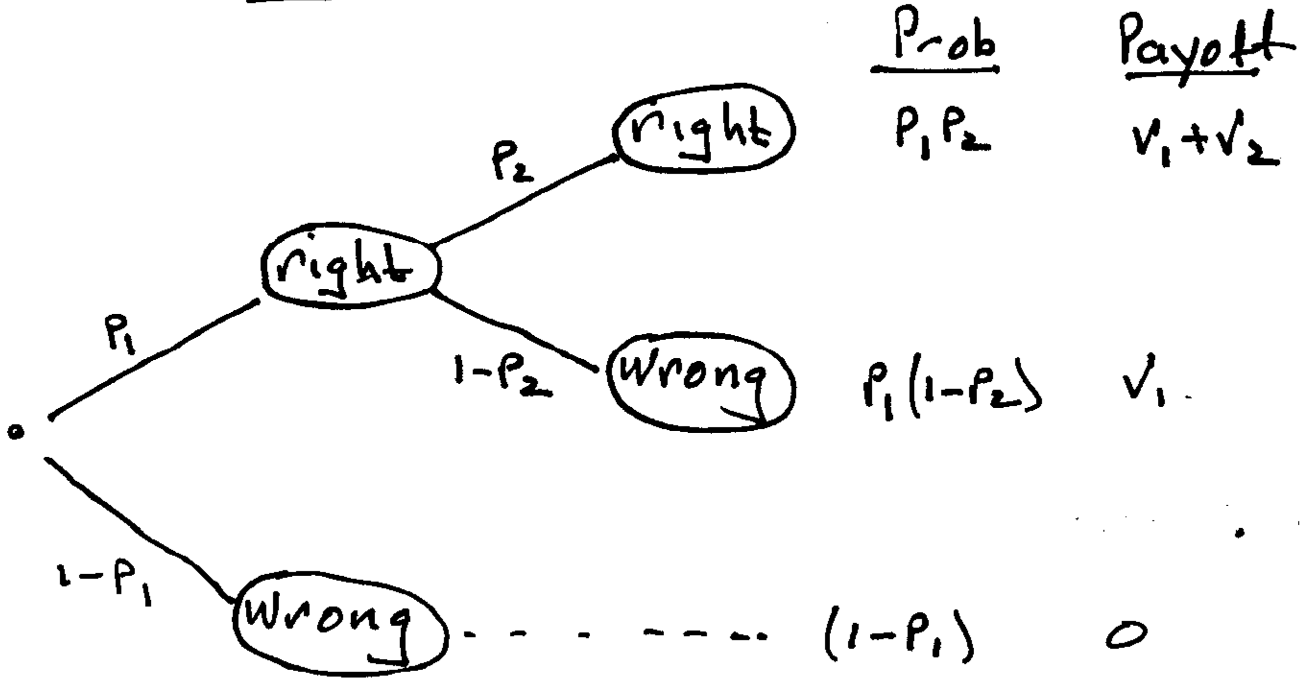
	Q_1	Q_2	
P(correct)	p_1	p_2	(independent)
Payoff in \$	v_1	v_2	

The 1st wrong answer terminates game. You can answer in either order: (Q_1, Q_2) or (Q_2, Q_1). Let X be the total payoff.

Suppose you choose order (Q_1, Q_2)

- Find PMF of X
- Find $E[X]$

Tree diagram



so

$$P_X(x) = \begin{cases} p_1 p_2 & \text{if } x = v_1 + v_2 \\ p_1(1-p_2) & \text{if } x = v_1 \\ 1-p_1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E[X] = (v_1+v_2)\cdot p_1 p_2 + v_1 p_1 (1-p_2) + 0\cdot(1-p_1)$$

$$= v_1 p_1 + v_2 p_1 p_2$$

Let Y be payoff in other order (Q_2, Q_1). Thus

$$P_Y(y) = \begin{cases} p_1 p_2 & \text{if } \quad y = v_1 + v_2 \\ p_2(1-p_1) & \quad y = v_2 \\ 1 - p_2 & \quad y = 0 \\ 0 & \quad \text{otherwise} \end{cases}$$

and

$$E[Y] = v_2 p_2 + v_1 p_1 p_2$$

We choose order (Q_1, Q_2) iff

$$E[X] > E[Y]$$

i.e. $\quad v_1 p_1 + v_2 p_1 p_2 > v_2 p_2 + v_1 p_1 p_2$

$$\therefore \quad V_1 P_1 - V_1 P_1 P_2 > V_2 P_2 - V_2 P_1 P_2$$

$$\therefore \quad V_1 P_1 (1 - P_2) > V_2 P_2 (1 - P_1)$$

$$\therefore \quad \boxed{\frac{V_1 P_1}{1 - P_1} > \frac{V_2 P_2}{1 - P_2}}$$

<u>Exercise</u>

Analyze this problem with 3 questions Q_1, Q_2, Q_3, i.e. 6 possible orders: 123, 132, 213, 231, 312, 321.

2.5 Joint PMFs and multiple r.v.s

Defn

let X, Y be discrete r.v.s.

Their joint PMF is

$$p_{X,Y}(x,y) = \underbrace{P(X=x, Y=y)}_{\text{shorthand for}}$$

$$P(\{X=x\} \cap \{Y=y\})$$

The joint PMF $p_{X,Y}(x,y)$ can be used to compute probabilities of events defined in terms of X and Y.

For instance, if

$$S \subseteq \text{range}(X) \times \text{range}(Y)$$

then

$$P((X,Y) \in S) = \sum_{(x,y) \in S} p_{X,Y}(x,y)$$

<u>Ex.</u> suppose $\text{range}(X) = \{1, 2, 3, 4\}$ and $\text{range}(Y) = \{1, 2, 3\}$. Let joint PMF is

$p_X(x)$:

4/20	5/20	6/20	5/20

$p_{X,Y}(x,y)$ and $p_Y(y)$:

$Y \backslash X$	1	2	3	4	$p_Y(y)$
3	1/20	1/20	2/20	1/20	5/20
2	2/20	3/20	3/20	1/20	9/20
1	1/20	1/20	1/20	3/20	6/20

suppose $Y=3$. since

$$\{X=1\}\cup\{X=2\}\cup\{X=3\}\cup\{X=4\}=\Omega$$

the total prob. thm we have

$$P(Y=3)=P(X=1,Y=3)+P(X=2,Y=3)$$
$$+P(X=3,Y=3)+P(X=4,Y=3)$$

i.e.

$$P_Y(3)=P_{X,Y}(1,3)+P_{X,Y}(2,3)+P_{X,Y}(3,3)+P_{X,Y}(4,3)$$

$$=\frac{1}{20}+\frac{1}{20}+\frac{3}{20}+\frac{1}{20}=\frac{5}{20}$$

In general:

$$\boxed{P_Y(y)=\sum_x P_{X,Y}(x,y)}$$

marginal PMF

we can recover P_X and P_Y from $P_{X,Y}$

$$\boxed{P_X(x) = \sum_y P_{X,Y}(x,y)}$$ marginal

$$\boxed{P_Y(y) = \sum_x P_{X,Y}(x,y)}$$ marginal.

<u>Functions of 2 r.v.s</u>

Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and let

$$Z = g(X,Y)$$

Then

$$P_Z(z) = \sum_{\{(x,y) \mid g(x,y)=z\}} P_{X,Y}(x,y)$$

This an incarnation of total Prob. thm.

Expected Value Rule (2 variables)

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot P_{X,Y}(x,y)}$$

Ex. show that

$$E[aX+bY+c] = aE[X]+bE[Y]+c$$

Proof.

$$E[aX+bY+c] = \sum_x \sum_y (ax+by+c) p_{X,Y}(x,y)$$

$$= \sum_x \sum_y ax\, p_{X,Y}(x,y) + \sum_x \sum_y by\, p_{X,Y}(x,y)$$

$$+ \sum_x \sum_y c\, p_{X,Y}(x,y)$$

$$= a \sum_x x \cdot \underline{\sum_y p_{X,Y}(x,y)} + b \sum_y y \cdot \underline{\sum_x p_{X,Y}(x,y)}$$

$$+ c \sum_x \sum_y p_{X,Y}(x,y)$$

$$= a\underbrace{\sum_x x\cdot p_X(x)} + b\underbrace{\sum_y y\cdot p_Y(y)} + c$$

$$= a\,E[X] + b\,E[Y] + c. \quad \blacksquare$$

we can generalize to any number of r.v.'s, say 3: X, Y, Z

$$p_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$$

2-way marginals

$$p_{X,Y}(x,y) = \sum_z p_{X,Y,Z}(x,y,z)$$

$$p_{X,Z}(x,z) = \cdots$$

$$p_{Y,Z}(y,z) = \cdots$$

Expected value rule

$$E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z) p_{X,Y,Z}(x,y,z)$$

suppose we have n random variables

$$X_1, X_2, \ldots, X_n$$

Then

$$\boxed{\begin{aligned} &E[a_1X_1 + a_2X_2 + \cdots + a_nX_n + b] \\ &= a_1E[X_1] + a_2E[X_2] + \cdots + a_nE[X_n] + b \end{aligned}}$$

The mean of the Binomial

Let X be Binomial(n,p), so

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k=0,1,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

The mean of X is

$$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k \cdot (1-p)^{n-k}$$

<u>Exercise</u> Prove sum is $\boxed{np}$

<u>Easy Way</u>

Observe $X = X_1 + X_2 + \cdots + X_n$ where X_i is Bernoulli(p). Recall

$$E[X_i] = p$$

Thus

$$E[X] = E[X_1 + X_2 + \cdots + X_n]$$
$$= E[X_1] + E[X_2] + \cdots + E[X_n]$$
$$= p + p + \cdots + p$$
$$= \boxed{np}$$

We will show that

$$Var(X) = \boxed{np(1-p)}$$

Ex. The hat problem

Suppose n people throw their hats into a box, then each picks a hat at random from box. The we have a random permutation, i.e. a mapping

$$\{\text{hats}\} \longrightarrow \{\text{people}\}$$

Fact: all $n!$ permutations are equally likely.

Let X = # people who get their own hat back. Find $E[X]$.

observe

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i is Bernoulli with $\boxed{p = \frac{1}{n}}$.

$$X_i = \begin{cases} 1 & \text{if person } i \text{ gets own hat back} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$E[X] = E[X_1 + X_2 + \cdots + X_n]$$
$$= E[X_1] + \cdots + E[X_n]$$
$$= \frac{1}{n} + \cdots + \frac{1}{n}$$
$$= n \cdot \frac{1}{n} = \boxed{1}$$

2.6 Conditioning

Given a r.v. $X : \Omega \to \mathbb{R}$, any valid Prob. law defines a PMF

$$p_X(x) = P(X = x)$$

This includes conditional Prob. law $P(\cdot \mid A)$ where $A \subseteq \Omega$ and $P(A) > 0$.

Defn

Let X be a r.v and $P(A) > 0$. The conditional PMF of X given A

is

$$\boxed{\begin{aligned} p_{X|A}(x) &= P(\{X=x\}|A) \\ &= \frac{P(\{X=x\}\cap A)}{P(A)} \end{aligned}}$$

Recall the events $\{X=x\}$ are disjoint for different x and, have union Ω

- $\{X=1\}\cap\{X=2\}=\phi$
- $\bigcup_{x\in \text{range}(X)} \{X=x\} = \Omega$

Thus $\{X=x\}$ (for $x\in\text{range}(X)$) form a partition of Ω. So

$\{X=x\}\cap A$ partition A

so by total Probability

$$P(A) = \sum_x P(\{X=x\} \cap A)$$

Thus

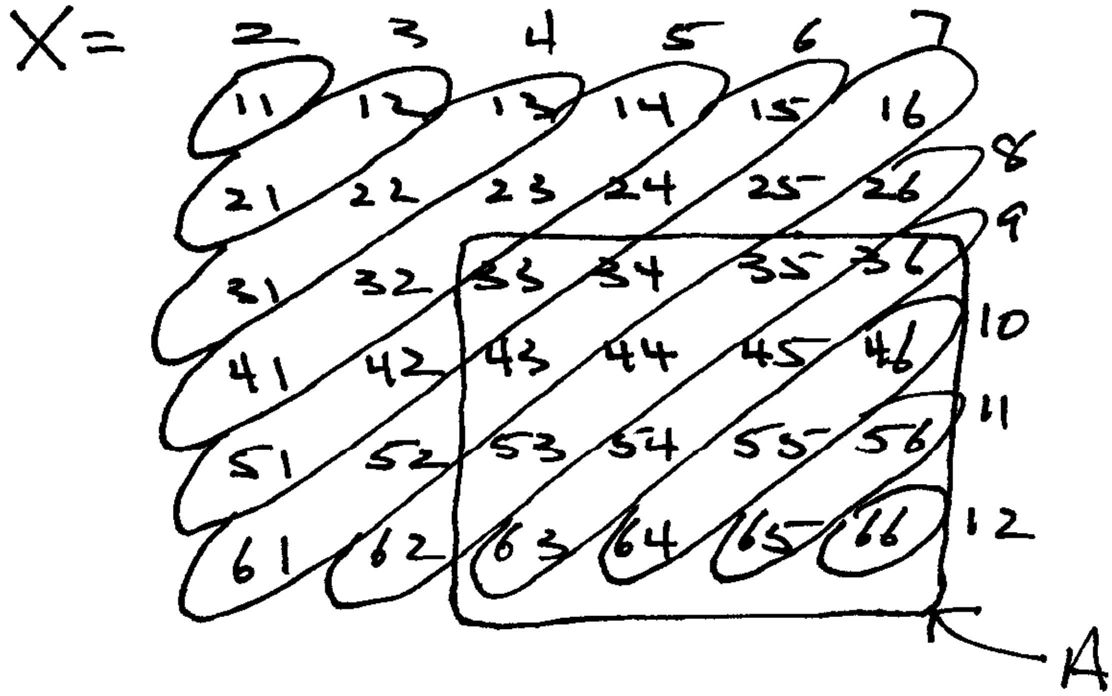
$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X=x\} \cap A)}{P(A)}$$

$$= \frac{1}{P(A)} \cdot \sum_x P(\{X=x\} \cap A)$$

$$= \frac{1}{P(A)} \cdot P(A) = 1$$

Ex.

Roll 2 indep., fair, 6-sided dice. Let X be the sum of the top faces.



$$P_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & k = 3, 11 \\ 3/36 & k = 4, 10 \\ 4/36 & k = 5, 9 \\ 5/36 & k = 6, 8 \\ 6/36 & k = 7 \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{\min \geq 3\}$, so $P(A) = \frac{16}{36}$.

$$P_{X|A}(k) = \begin{cases} 1/16 & \text{if } k = 6, 12 \\ 2/16 & \text{if } k = 7, 11 \\ 3/16 & \text{if } k = 8, 10 \\ 4/16 & \text{if } k = 9 \\ 0 & \text{otherwise} \end{cases}$$

<u>Ex</u>

Flip a weighted coin with $P(H) = p$ until 1st head. Let X be the # of flips. so X is Geometric(p)

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{1^{st}$ head within n flips$\}$

$= \{X \leq n\}$. so

$$P(A) = \sum_{k=1}^{n} P_X(k)$$

$$= \sum_{k=1}^{n} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=1}^{n} (1-p)^{k-1} \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= p \cdot \sum_{k=0}^{n-1} (1-p)^{k}$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}}$$

$$= 1-(1-p)^n$$

Another way

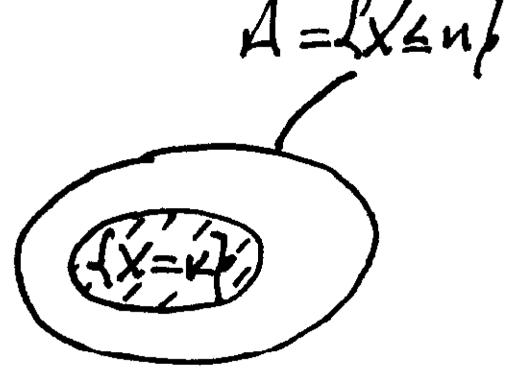
$$P(A) = 1 - P(A^c)$$

$$= 1 - P(1^{st}\ n \text{ flips are tails})$$

$$= 1 - (1-p)^n$$

If $1 \le k \le n$, then

$$P_{X|A}(k) = \frac{P(\{X=k\} \cap A)}{P(A)}$$



$$= \frac{P(X=k)}{P(A)}$$

$$= \frac{(1-p)^{k-1} \cdot p}{1-(1-p)^n}$$

so

$$P_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1} \cdot p}{1-(1-p)^n} & \text{if } k = 1, 2, \ldots n \\ 0 & \text{otherwise} \end{cases}$$

Now let Y another r.v. with

$$p_Y(y) = P(Y=y) > 0$$

for $y \in \underline{\text{range}}(Y)$. We specialize A to be $A = \{Y=y\}$.

<u>Defn</u>

The conditional PMF of X given Y is

$$\boxed{\begin{aligned} p_{X|Y}(x|y) &= P(X=x \mid Y=y) \\ &= \frac{P(X=x,\, Y=y)}{P(Y=y)} \end{aligned}}$$

equivalently

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

Thus

$$\boxed{P_{X,Y}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)}$$

and

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)}$$

Ex. (Professor problem)

A Professor answers wrongly with Prob. $1/4$, indep. of other questions. The # of questions asked is 0, 1 or 2 with equal. Prob. $1/3$.

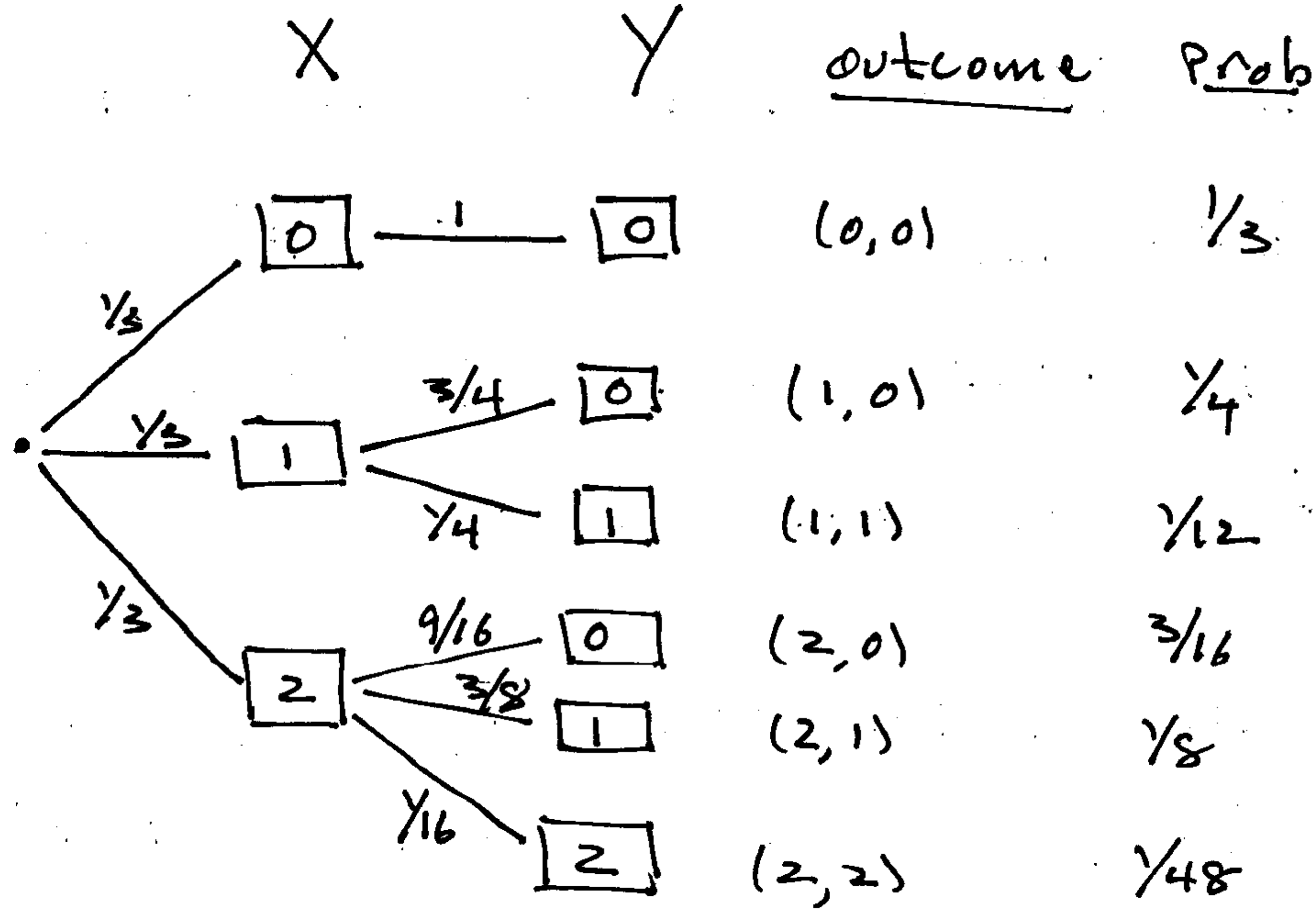
Let X = # questions asked, and Y = # questions answered wrongly.

- Determine joint PMF $p_{X,Y}(x,y)$.

we have

$$p_X(x) = \begin{cases} 1/3 & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

tree diagram



$$P(Y=0 \mid X=0) = 1$$

$$P(Y=1 \mid X=1) = \frac{1}{4}$$

$$P(Y=0 \mid X=1) = \frac{3}{4}$$

$$P(Y=0 \mid X=2) = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}$$

$$P(Y=1 \mid X=2) = \frac{1}{4}\cdot\frac{3}{4}+\frac{3}{4}\cdot\frac{1}{4} = 2\cdot\frac{3}{16} = \frac{3}{8}$$

$$P(Y=2 \mid X=2) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$$

#asked #wrong

Thus

$$p_{Y|X}(y|x) = \begin{cases} 1 & x=0,\ y=0 \\ 3/4 & x=1,\ y=0 \\ 1/4 & x=1,\ y=1 \\ 9/16 & x=2,\ y=0 \\ 3/8 & x=2,\ y=1 \\ 1/16 & x=2,\ y=2 \\ 0 & \text{otherwise} \end{cases}$$

and therefore

$$p_{X,Y}(x,y) = \begin{cases} 1/3 & x=0,\ y=0 \\ 1/4 & x=1,\ y=0 \\ 1/12 & x=1,\ y=1 \\ 3/16 & x=2,\ y=0 \\ 1/8 & x=2,\ y=1 \\ 1/48 & x=2,\ y=2 \\ 0 & \text{otherwise} \end{cases}$$

can now find the other marginal $p_Y(y)$, and the other conditional $p_{X|Y}(x|y)$

as tables!

$p_X(x)$

x	0	1	2
	$1/3$	$1/3$	$1/3$

$p_{Y|X}(y|x)$

y \ x	0	1	2
2			$1/16$
1		$1/4$	$3/8$
0	1	$3/4$	$9/16$

$p_{X,Y}(x,y)$

y \ x	0	1	2	$p_Y(y)$
2			$1/48$	$1/48$
1		$1/12$	$1/8$	$10/48$
0	$1/3$	$1/4$	$3/8$	$37/48$

Now find $P_{X|Y}(x|y) = \dfrac{P_{X,Y}(x,y)}{P_Y(x)}$

$P_{X|Y}(x|y)$

$y \backslash x$	0	1	2
2			1
1		3/5	3/5
0	16/37	12/37	9/37

Conditional Expectation

Let X be a r.v., $A \subseteq \Omega$ with $P(A) > 0$. Define

$$\boxed{E[X|A] = \sum_x x\, P_{X|A}(x)}$$

also

$$\boxed{E[g(X) \mid A] = \sum_x g(x) \cdot p_{X|A}(x)}$$

we can condition on $A = \{Y = y\}$ provided $P(Y=y) > 0$.

$$\boxed{E[X \mid Y=y] = \sum_x x \, p_{X|Y}(x|y)}$$

Theorem (total expectation)

Let $A_1, \ldots, A_n$ form a partition of Ω, then

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]}$$

<u>Proof</u>

By total Probability

$$p_X(x) = \sum_{i=1}^{n} P(A_i)\, p_{X|A_i}(x)$$

multiply both sides by x, and sum over $x \in \text{range}(X)$:

$$E[X] = \sum_x x\, p_X(x)$$

$$= \sum_x x \cdot \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

$$= \sum_x \sum_{i=1}^{n} P(A_i) \cdot x\, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \sum_x x\, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]$$

$\blacksquare$